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Interaction of phase-locked modes: a new mechanism for the rapid growth of three-dimensional disturbances

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In this paper, we have identified a new mechanism which can promote rapid growth of three-dimensional disturbances. The mechanism involves the interaction between a planar mode and an oblique mode, or a pair of oblique modes, which are *phase-locked* in the sense that they have the same phase speed. This allows a powerful nonlinear interaction to take place within the common critical layer(s). The disturbance is not required to form a subharmonic resonant triad, and hence the mechanism operates under much less restrictive conditions than does subharmonic resonance (although it is somewhat less powerful). We show that the quadratic interaction between the planar mode and the oblique modes drives an exceptionally large forced mode with the difference frequency, which in turn interacts with the planar mode to contribute the dominant nonlinear effect. This interaction can cause the oblique modes to grow super-exponentially provided that their magnitude is sufficiently small. As a result of the super-exponential growth, the oblique mode may soon become strong enough to produce a feedback effect on the planar mode, so that the interactions eventually become fully coupled. This subsequent stage takes slightly different forms depending on whether a single or a pair of oblique modes is present. Both cases are investigated. Particular attention is paid to symmetric plane shear layers, e.g. a planar wake or jet, for which subharmonic resonance of sinuous modes is inactive.

1. Introduction

In an incompressible parallel or nearly parallel flow, laminar-turbulent transition is often initiated by a predominantly two-dimensional disturbance, which develops according to linear instability theory. However, sufficiently far downstream, three-dimensional disturbances grow more rapidly, and become dominant. This is presumably a nonlinear effect since linear theory predicts that the fastest-growing mode is two-dimensional.

At least two mechanisms have been suggested to explain this rapid growth of three-dimensional disturbances. The first is the '*secondary-instability*' theory (see e.g. Herbert 1988 and references therein), which proposes that the mean flow plus a finite-amplitude two-dimensional disturbance can support three-dimensional instability waves. While this is an attractive and physically valid notion, it still lacks a solid mathematical justification except in the case where an equilibrium two-dimensional wave exists.

The second mechanism is the subharmonic *resonant-triad* interaction proposed by Raetz (1959) and Craik (1971) for boundary-layer transition. Recent studies of subharmonic resonance have given considerable insight into transition processes in boundary layers and other shear flows: see for example Goldstein & Lee (1992), Wu (1992, 1993*b*, 1995*a*), Mankbadi, Wu & Lee (1993), Wundrow, Hultgren & Goldstein (1994) and Lee (1994). However, some important transition phenomena cannot be explained by such a mechanism, of which the following are examples.

(*a*) Disturbances which do not form a resonant triad are also observed to develop quickly. For instance, Corke & Mangano (1989) find that in addition to the subharmonic, a broad band of three-dimensional disturbances also undergoes enhanced growth.

(*b*) A symmetric shear flow such as a plane wake supports two types of linear instability modes, namely 'sinuous' modes, for which the vertical velocity is symmetric about the centreline, and 'varicose' modes, for which it is antisymmetric. Subharmonic resonance can take place among varicose modes (Mallier 1995), but for sinuous modes such a mechanism is inactive because the contributions from the two critical layers cancel (Wu 1995*a*). On the other hand, linear theory predicts that the fastest-growing instabilities are sinuous modes, and indeed these are observed to dominate in the early stages of transition. It is not clear, therefore, that subharmonic resonance can convincingly explain the rapid amplification of three-dimensional disturbances observed in a plane wake (Corke, Krull & Ghassemi 1992; Williamson & Prasad 1993*a*,*b*).

This paper offers a new explanation for such observations. The proposed mechanism involves a nonlinear interaction between a planar mode and an oblique mode (or a pair of oblique modes) which have the same phase speed, and hence share the same critical layer(s). We shall refer to this as a *phase-locked modal interaction*. The mechanism can operate in any quasi-parallel flow, symmetric or asymmetric, which admits Rayleigh instabilities. For an asymmetric flow, or for varicose modes in a symmetric flow, we shall assume that the disturbance does not form a subharmonic resonant triad; the reason for which will be explained in §5.3. For sinuous modes in a symmetric flow, such a restriction is unnecessary. In both cases, we show that the interaction contributes a cubic term in the amplitude equation, which can cause the oblique modes to grow super-exponentially, while the planar mode is still evolving exponentially, unaffected by the oblique modes. This is very similar to the effect of the usual quadratic parametric resonance (see e.g. Goldstein & Lee 1992). Eventually the oblique modes may become strong enough to produce a feedback effect on the planar mode through mutual interactions at the cubic level, and the development of the modes is then 'fully coupled'. In addition, if a pair of oblique modes is introduced, the self-interaction between the oblique modes also comes into play at this stage.

The paper is organized as follows. In the next section, we formulate the problem for an arbitrary shear-flow profile which is assumed to be inviscidly unstable. The underlying asymptotic scalings are derived and explained, and an important interplay between the flows inside and outside the critical layers is illustrated. In §3, we perform an appropriate asymptotic expansion of the solution in the main part of the flow. In §4, the flow within the viscous, non-equilibrium critical layers is analysed and solutions are found analytically. Matching them onto those in the outer region, we obtain the (coupled) amplitude equations (§5). In §6, the generic analysis is specialized to a plane wake, and the coefficients involved in the amplitude equations are evaluated explicitly. The amplitude equations are studied in §7, both analytically and numerically. In §8, the results are discussed and related to experiments on transition in a plane wake.

2. Formulation and scalings

We assume that the flow is described in terms of Cartesian coordinates (x, y, z), where x, y and z are streamwise, transverse and spanwise coordinates respectively; they are non-dimensionalized by δ_*, the thickness of the shear layer at a typical streamwise location, say $x = 0$. The time (t), velocity and pressure are non-dimensionalized by δ_*/U_*, U_* and $\rho_* U_*^2$ respectively, where U_* is a reference velocity and ρ_* is the density of the fluid. The Reynolds number R is defined as

$$R = \frac{U_* \delta_*}{\nu_*} , \tag{2.1}$$

where ν_* is the kinematic viscosity. Throughout this paper, we shall assume that $R \gg 1$ so that a self-consistent theory can be constructed. The analysis applies to any inviscidly unstable, quasi-parallel two-dimensional flow with velocity profile

$$(\bar{U}(x_3, y), R^{-1}\bar{V}(x_3, y), 0) , \tag{2.2}$$

where the dependence on the slow variable

$$x_3 = x/R \tag{2.3}$$

represents the non-parallel flow effect. We denote the perturbed flow by

$$(\bar{U} + u, R^{-1}\bar{V} + v, w) . \tag{2.4}$$

The following two types of disturbance will be investigated:

Case I: a planar mode (α_0, 0, c_0) with magnitude of order ϵ interacts with a *single* oblique mode (α, $-\beta$, c) with magnitude of order δ, where α_0 is the streamwise wavenumber of the planar mode and c_0 its phase speed, while α, β and c represent respectively the streamwise and spanwise wavenumbers and the phase speed of the oblique mode.

Case II: a planar mode (α_0, 0, c_0) interacts with pair of oblique modes (α, $\pm\beta$, c). It turns out that the relevant amplitude equations for this case can be obtained by a straightforward modification of those for case I.

Both the planar and the oblique modes satisfy Rayleigh's equation to leading order. According to linear theory, a mode excited upstream will grow exponentially, and eventually becomes neutral at some downstream location, because of the viscous spreading of the shear layer. It follows from Squire's transformation that if

$$\alpha^2 + \beta^2 = \alpha_0^2 , \tag{2.5}$$

the planar mode $(\alpha_0, 0, c_0)$ and the oblique modes $(\alpha, \pm\beta, c)$ all become neutral at the same location. Nonlinear effects are likely to come into play near such a point since according to linear theory the disturbance attains its maximum amplitude there, and because, as will become apparent, the nonlinearity can come into play at much smaller magnitudes in this region. Moreover, when condition (2.5) is satisfied, the modes are 'phase-locked' in the sense that they have the same phase speed, i.e.

$$c_0 = c , \tag{2.6}$$

and so share the same critical layer(s). This fact is significant because it allows for particularly efficient nonlinear interactions to take place in the common critical layer(s).

Condition (2.5) can be represented in wavenumber space as shown in figure 1. For a given α_0, *any* point on the semicircle represents an oblique mode which has the same

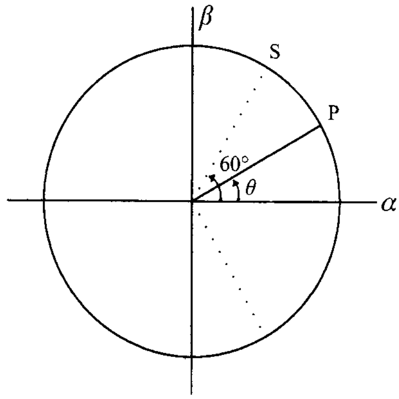

FIGURE 1. Conditions for the subharmonic resonance and the phase-locked interaction. For a given α_0, any point on the semicircle, $\alpha^2 + \beta^2 = \alpha_0^2$ ($\alpha > 0$), represents an oblique mode which is phase-locked with the planar mode $(\alpha_0, 0, c)$. The subharmonic resonance occurs for $\theta \approx 60°$, while the phase-locked interaction occurs for virtually all θ.

phase velocity as the planar mode, and hence can participate in the phase-locked interaction. Subharmonic resonance is a special case corresponding to $\theta = 60°$. We note, however, that (2.5) is not always a necessary condition: it is merely required that the modes are close to neutral and satisfy (2.6). For example, in a symmetric plane wake or jet, a phase-locked interaction can take place between a (nearly) neutral sinuous mode and a (nearly) neutral varicose mode since they have the same phase speed, even though (2.5) is not satisfied. Therefore, we shall not restrict our analysis to (2.5). We shall always assume, however, that

$$\alpha_0 > \alpha \,, \tag{2.7}$$

which is certainly the case when (2.5) holds; the reason for this assumption is explained below (2.15).

For the reasons stated earlier, it is thus appropriate to assume that nonlinear effects are important in the vicinity of the neutral location, where the local (spatial) growth rate is small, of order μ say. It follows that the local frequencies of the plane and oblique modes, ω_0 and ω, deviate from their local neutral values by $O(\mu)$. More precisely

$$\omega_0 = \alpha_0(c - \mu S_0) \,, \quad \omega = \alpha(c - \mu S) \,, \tag{2.8}$$

where S_0 and S are positive and of order one. For simplicity, in this paper we assume $S_0 = S$; the more general situation with $S_0 \neq S$ can be analysed by a straightforward extension. It is then convenient to introduce

$$\zeta = x - (c - \mu S)t \,, \tag{2.9}$$

which is the coordinate travelling with the common phase speed. To leading order, the vertical velocity of the planar mode has the form

$$v_{2D} \sim \epsilon B(x_1)\phi_b(y)\mathrm{e}^{\mathrm{i}\alpha_0\zeta} + c.c. + \dots \,, \tag{2.10}$$

where $\epsilon \ll 1$, and B is of order one and is a function of the intermediate spatial variable

$$x_1 = \mu c^{-1} x \,. \tag{2.11}$$

(The factor c^{-1} is introduced for later convenience.)

For the three-dimensional disturbance

$$v_{3D} \sim \delta A(x_1)\phi_a(y)\mathrm{e}^{\mathrm{i}(\alpha\zeta-\beta z)} + c.c. + \dots \, . \tag{2.12}$$

In the present paper, it is assumed that both the planar and the oblique wave grow in the streamwise direction. This would be the case for an initial excitation periodic along the spanwise direction (see e.g. Schneider & Coles 1994). However, for other types of excitation, it may be appropriate to allow for a more general modulation by introducing a slow spanwise variable together with the streamwise variable x_1.

As in Wu (1992), in order to derive the underlying scaling, we write down the x-momentum equation for the disturbance, namely

$$\left[\mu c^{-1}\bar{U}\frac{\partial}{\partial x_1} + (\bar{U} - c + \mu S)\frac{\partial}{\partial \zeta}\right] u + \bar{U}'v - R^{-1}\Delta u = -\frac{\partial p}{\partial x} - \frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} - \frac{\partial uw}{\partial z}, \tag{2.13}$$

where $\partial/\partial t$ has been replaced by $-(c - \mu S)\partial/\partial\zeta$ in view of (2.9). We are interested in the so-called non-equilibrium critical-layer régime, in which the spatial variation term $\mu c^{-1}\bar{U}\partial u/\partial x_1$ is comparable with $(\bar{U} - c + \mu S)\partial u/\partial\zeta$. This implies that the thickness of the critical layer is of $O(\mu)$. As a critical level, $y = y_c$ say, is approached, the streamwise velocity of the oblique mode exhibits a simple pole singularity, $u_{3D} \sim \delta(y - y_c)^{-1}$, so that $u_{3D,y} = O(\delta\mu^{-2})$ in the critical layer. The normal velocities of the planar and oblique waves are of order ϵ and δ respectively, both within the critical layers and in the main part of the flow. Within a critical layer, the interaction between the planar and the oblique modes, typically through the $(\partial/\partial y)(v_{2D}u_{3D})$ term, produces a forcing term of $O(\epsilon\delta\mu^{-2})$ in the x-momentum equation. This generates two forced modes $(\alpha_0 \pm \alpha, \mp\beta, c)$, both of which have a u-component of magnitude of $O(\epsilon\delta\mu^{-3})$, as can be deduced by balancing the forcing with the $\mu c^{-1}\bar{U}\partial u/\partial x_1$ term in (2.13). We find that the streamwise velocity of the sum mode $(\alpha_0 + \alpha, -\beta, c)$ decays (algebraically) towards the edge of the critical layer, while that of the difference mode $(\alpha_0 - \alpha, \beta, c)$ does not, and thus has a magnitude of $O(\epsilon\delta\mu^{-3})$ in the main part of the flow. It follows that the difference-mode pressure is of the same order, and so drives a streamwise velocity, say u_d, of $O(\epsilon\delta\mu^{-4})$ in the critical layer. The interaction of this with the planar mode, typically through the $(\partial/\partial y)(v_{2D}u_d)$ term, generates a u-component of $O(\epsilon^2\delta\mu^{-6})$ in the oblique mode $(\alpha, -\beta, c)$. Nonlinearity will affect the evolution of the oblique mode if this nonlinearly generated velocity balances the $O(\delta\mu)$ velocity discontinuity in the solution in the outer region, i.e. when $\epsilon^2\delta\mu^{-6} \sim \delta\mu$. This implies the distinguished scaling

$$\epsilon = \mu^{7/2}\, . \tag{2.14}$$

Once the planar mode has attained this magnitude, it will affect the development of the oblique mode even when the latter is infinitesimal. As will be shown later, provided that the oblique mode is not too large ($\delta \ll O(\epsilon^{6/7})$), the planar mode still evolves linearly, unaffected by the presence of the oblique mode. We refer to this régime as the 'secondary-instability' stage. It turns out that in this stage, the oblique mode can experience super-exponential growth.

However if the magnitude of the oblique mode becomes sufficiently large, either through initial forcing or as a result of the super-exponential growth, it will produce a back reaction on the planar mode. It turns out that the interaction between the oblique mode and the leading-order difference mode does not produce a velocity jump across the critical layer, and so does not influence the evolution of the planar-mode amplitude. The dominant feedback comes from the interaction of the oblique mode with the *second-order* difference mode and the sum mode, both of which are of $O(\epsilon\delta\mu^{-3})$ within the critical layer, to produce an $O(\epsilon\delta^2\mu^{-4})$ forcing. This generates an $O(\epsilon\delta^2\mu^{-5})$ correction in the planar mode, which balances the linear growth when $\epsilon\delta^2\mu^{-5} \sim \epsilon\mu$. The appropriate scaling is thus

$$\delta = \mu^3 = \epsilon^{6/7} . \tag{2.15}$$

We note that this asymmetry between the scalings for the planar and oblique modes depends on the assumption (2.7). If instead $\alpha_0 < \alpha$, the roles of the planar and oblique modes would be exchanged: the planar mode could not influence the oblique mode until its amplitude reached $O(\mu^3)$, while the oblique mode would require a magnitude of only $O(\mu^{7/2})$ to affect the planar mode. Since in the linear régime the fastest-growing mode is two-dimensional, it seems likely that a planar mode will be the first to reach the necessary magnitude, $O(\mu^{7/2})$, to provoke a nonlinear interaction of this type. The assumption (2.7) is based on this consideration.

In the following, we shall concentrate on the 'fully-coupled' stage, where the planar and oblique modes have magnitudes specified by (2.14) and (2.15). The 'secondary-instability' régime, corresponding to $\delta \ll O(\epsilon^{6/7})$, can be obtained in an appropriate limit (see §7.2). When a pair of oblique modes is included, rather than a single oblique mode, the fully coupled stage takes a slightly different form since the interaction between the oblique modes contributes an extra nonlinear term to the final amplitude equations. It should be emphasized, however, that the phase-locked mechanism is a *two-mode* interaction, rather than a *three-mode* interaction as for subharmonic or side-band resonance. In order to illustrate this, the detailed analysis will be carried out for a planar wave and a single oblique wave. The appropriate amplitude equations for a planar wave and a pair of oblique waves can then be obtained by incorporating the results of Wu, Lee & Cowley (1993) for a pair of oblique modes alone.

In deriving the above scaling, we have assumed that the non-equilibrium effect represented by $\partial/\partial x_1$ appears at leading-order in the critical-layer equations. On the other hand, the viscous term, $R^{-1}u_{yy}$, is also a leading-order effect when it balances $(\bar{U} - c + \mu)\partial u/\partial\zeta$. This occurs when

$$R^{-1} = \lambda\mu^3 , \tag{2.16}$$

where the 'Haberman parameter' λ is of order one (cf. Haberman 1972; Wu *et al.* 1993). The critical layer is then both non-equilibrium and viscous in nature. The inviscid limit, $R \to \infty$, can be immediately obtained by setting $\lambda = 0$ in the analysis, while the strongly viscous case, $\lambda \gg 1$, is briefly discussed in §7.1.

The basic flow $\bar{U}$ evolves on the very slow streamwise scale x_3, and it is sufficient to approximate its profile at x_3 by a Taylor expansion about $x_3 = 0$:

$$\bar{U}(x_3, y) = \bar{U}(0, y) + x_3\frac{\partial\bar{U}}{\partial x_3}(0, y) + \ldots = \bar{U}(0, y) + \lambda\mu^2 c x_1\frac{\partial\bar{U}}{\partial x_3}(0, y) + \ldots . \tag{2.17}$$

In the following, it is understood that all mean-flow quantities are evaluated at $x_3 = 0$, and that a prime denotes differentiation with respect to y.

3. Outer expansion and solution

Outside the critical layers, the flow perturbation is predominantly linear and inviscid. The pressure p and velocity (u, v, w) of the disturbance are expanded as

$$\left.\begin{aligned} p &= \delta\,[p_{a1} + \mu p_{a2} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon\,[p_{b1} + \mu p_{b2} + \ldots]\,\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\ &\quad + \delta\epsilon\mu^{-3}\,[p_{d1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \delta\epsilon\mu^{-1}\,[p_{s1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} + c.c. + \ldots\,, \\ u &= \delta\,[u_{a1}(x_1, y) + \mu u_{a2} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon\left[\mathrm{i}\alpha_0^{-1}B(x_1)\phi_b'(y) + \mu u_{b2} + \ldots\right]\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\ &\quad + \delta\epsilon\mu^{-3}\,(u_{d1} + \ldots)\,\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \delta\epsilon\mu^{-1}\,(u_{s1} + \ldots)\,\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} + c.c. + \ldots\,, \\ v &= \delta\,[A(x_1)\phi_a(y) + v_{a2} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon\,[B(x_1)\phi_b(y) + \mu v_{b2} + \ldots]\,\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\ &\quad + \delta\epsilon\mu^{-3}\,[v_{d1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \delta\epsilon\mu^{-1}\,[v_{s1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} + c.c. + \ldots\,, \\ w &= \delta\,[w_{a1} + \mu w_{a2} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \delta\epsilon\mu^{-3}\,[w_{d1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \ldots \\ &\quad + \delta\epsilon\mu^{-1}\,[w_{s1} + \ldots]\,\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} + c.c. + \ldots\,, \end{aligned}\right\} \tag{3.1}$$

where

$$\alpha_d = \alpha_0 - \alpha\,, \quad \alpha_s = \alpha_0 + \alpha \tag{3.2}$$

are the streamwise wavenumbers of the forced difference and sum modes respectively. The amplitude parameters δ and ϵ are specified by (2.14) and (2.15), but are here retained in primitive form so that the meaning and the origin of different terms in the expansion become more apparent.

The eigenfunction $\phi_a(y)$ satisfies Rayleigh's equation

$$\phi_a'' - \left(\alpha^2 + \beta^2 + \frac{\bar{U}''}{\bar{U} - c}\right)\phi_a = 0\,, \tag{3.3}$$

and boundary conditions†

$$\phi_a \to 0 \quad \text{as} \quad y \to \pm\infty\,. \tag{3.4}$$

For the amplitude $A(x_1)$ to be precisely defined, it is also necessary to impose a normalization condition on ϕ_a, say $\phi_a(y_0) = 1$ for some specified y_0. Likewise, $\phi_b(y)$ satisfies

$$\phi_b'' - \left(\alpha_0^2 + \frac{\bar{U}''}{\bar{U} - c}\right)\phi_b = 0\,, \tag{3.5}$$

and the same boundary conditions as ϕ_a, plus a suitable normalization condition. As noted in the previous section, phase-locking is guaranteed when $\alpha^2 + \beta^2 = \alpha_0^2$, in which case we may take $\phi_b = \phi_a$. However, we shall proceed for the more general case.

Let $\eta = y - y_j$, where y_j is the jth critical level at which $\bar{U} = c$. Then as $\eta \to \pm 0$,

$$\phi_a \sim b_j^{\pm}[1 + p_j\eta\log|\eta|] + a_j^{\pm}\eta + O(\eta^2\log|\eta|)\,, \tag{3.6}$$

$$\phi_b \sim \tilde{b}_j^{\pm}[1 + p_j\eta\log|\eta|] + \tilde{a}_j^{\pm}\eta + O(\eta^2\log|\eta|)\,, \tag{3.7}$$

where $a_j^{\pm}$, $b_j^{\pm}$, $\tilde{a}_j^{\pm}$ and $\tilde{b}_j^{\pm}$ are constants, and

$$p_j = \frac{\bar{U}_j''}{\bar{U}_j'}\,, \quad \bar{U}_j' = \frac{\partial\bar{U}}{\partial y}(0, y_j)\,, \quad \text{etc.} \tag{3.8}$$

From an analysis of the critical layer (see the next section) it is readily shown that

$$b_j^+ = b_j^- \equiv b_j\,, \quad \tilde{b}_j^+ = \tilde{b}_j^- \equiv \tilde{b}_j\,. \tag{3.9}$$

† These boundary conditions are appropriate to a free shear layer. For a wall shear layer, we require $\phi_a(0) = 0$, $\phi_a \to 0$ as $y \to \infty$.

The oblique-mode correction v_{a2} satisfies the inhomogeneous Rayleigh equation

$$\frac{\partial^2 v_{a2}}{\partial y^2} - \left(\alpha^2 + \beta^2 + \frac{\bar{U}''}{\bar{U}-c}\right) v_{a2} = -\frac{2\,\mathrm{i}\alpha}{c}\frac{\mathrm{d}A}{\mathrm{d}x_1}\phi_a - \left[\frac{1}{\mathrm{i}\alpha}\frac{\mathrm{d}A}{\mathrm{d}x_1} + SA\right]\frac{\bar{U}''}{(\bar{U}-c)^2}\phi_a\,. \quad (3.10)$$

The planar-mode correction v_{b2} satisfies the same equation with α, β and A replaced by α_0, 0, and B respectively. As $\eta \to \pm 0$,

$$\begin{aligned} v_{a2} \sim\ & - b_j r_j(x_1)\log|\eta| + d_j^{\pm}(x_1) + [a_j^{\pm} r_j(x_1) + b_j s_j(x_1) + p_j d_j^{\pm}(x_1)]\eta\log|\eta| \\ & + c_j^{\pm}(x_1)\eta + O(\eta^2\log|\eta|)\,, \end{aligned} \quad (3.11)$$

$$\begin{aligned} v_{b2} \sim\ & - \tilde{b}_j \tilde{r}_j(x_1)\log|\eta| + \tilde{d}_j^{\pm}(x_1) + [\tilde{a}_j^{\pm} \tilde{r}_j(x_1) + \tilde{b}_j \tilde{s}_j(x_1) + p_j \tilde{d}_j^{\pm}(x_1)]\eta\log|\eta| \\ & + \tilde{c}_j^{\pm}(x_1)\eta + O(\eta^2\log|\eta|)\,, \end{aligned} \quad (3.12)$$

where $c_j^{\pm}$, $d_j^{\pm}$, $\tilde{c}_j^{\pm}$ and $\tilde{d}_j^{\pm}$ are yet to be determined, and

$$r_j = -\left[\frac{1}{\mathrm{i}\alpha}\frac{\mathrm{d}A}{\mathrm{d}x_1} + SA\right]\frac{\bar{U}_j''}{\bar{U}_j'^2}\,, \quad (3.13)$$

$$s_j = -\left[\frac{1}{\mathrm{i}\alpha}\frac{\mathrm{d}A}{\mathrm{d}x_1} + SA\right]\frac{\bar{U}_j'\bar{U}_j''' - \bar{U}_j''^2}{\bar{U}_j'^3}\,. \quad (3.14)$$

The expressions for $\tilde{r}_j$ and $\tilde{s}_j$ are the same as for r_j and s_j except that α and A are replaced by α_0 and B respectively. Multiplying both sides of (3.10) by ϕ_a, and integrating from $-\infty$ to ∞, we obtain the solvability condition for v_{a2}:

$$\sum_j \left[b_j(c_j^+ - c_j^-) - (a_j^+ d_j^+ - a_j^- d_j^-)\right] = \frac{2\,\mathrm{i}\alpha}{c}\frac{\mathrm{d}A}{\mathrm{d}x_1}J_1 + \left[\frac{1}{\mathrm{i}\alpha}\frac{\mathrm{d}A}{\mathrm{d}x_1} + SA\right]J_2\,. \quad (3.15)$$

Here the sum is over all critical layers; J_1 and J_2 are constants defined by

$$J_1 = \int_{-\infty}^{\infty}\phi_a^2\,\mathrm{d}y\,, \quad J_2 = \mathrm{FP}\int_{-\infty}^{\infty}\frac{\bar{U}''\phi_a^2}{(\bar{U}-c)^2}\,\mathrm{d}y\,, \quad (3.16)$$

and FP denotes the Hadamard finite part. Similarly, the corresponding equation for v_{b2} yields the solvability condition

$$\sum_j \left[\tilde{b}_j(\tilde{c}_j^+ - \tilde{c}_j^-) - (\tilde{a}_j^+ \tilde{d}_j^+ - \tilde{a}_j^- \tilde{d}_j^-)\right] = \frac{2\,\mathrm{i}\alpha_0}{c}\frac{\mathrm{d}B}{\mathrm{d}x_1}\tilde{J}_1 + \left[\frac{1}{\mathrm{i}\alpha_0}\frac{\mathrm{d}B}{\mathrm{d}x_1} + SB\right]\tilde{J}_2\,, \quad (3.17)$$

where

$$\tilde{J}_1 = \int_{-\infty}^{\infty}\phi_b^2\,\mathrm{d}y\,, \quad \tilde{J}_2 = \mathrm{FP}\int_{-\infty}^{\infty}\frac{\bar{U}''\phi_b^2}{(\bar{U}-c)^2}\,\mathrm{d}y\,. \quad (3.18)$$

The forced difference mode at $O(\epsilon\delta\mu^{-3})$ satisfies Rayleigh's equation

$$\frac{\partial^2 v_{d1}}{\partial y^2} - \left(\alpha_d^2 + \beta^2 + \frac{\bar{U}''}{\bar{U}-c}\right) v_{d1} = 0\,. \quad (3.19)$$

As $\eta \to \pm 0$, v_{d1} has the local asymptotic solution,

$$v_{d1} \sim D_j^{\pm}(x_1)[1 + p_j\eta\log|\eta|] + C_j^{\pm}(x_1)\eta + O(\eta^2\log|\eta|)\,, \quad (3.20)$$

and the critical-layer analysis gives

$$D_j^+ = D_j^- \equiv D_j\,. \quad (3.21)$$

Equation (3.19) is to be solved subject to the boundary conditions

$$v_{d1} \to 0 \quad \text{as} \quad y \to \pm\infty , \tag{3.22}$$

together with the inhomogeneous jump conditions (4.18) which will be derived in the next section. It is being tacitly assumed that the forced difference mode is not in resonance with an eigenmode of the system, so that the problem specified by (3.19)–(3.22) and (4.18) does indeed have a solution. A detailed discussion of this point is postponed to §5.3.

The leading-order pressure of the planar mode is given by

$$p_{b1} = \mathrm{i}\alpha_0^{-1}[\bar{U}'\phi_b - (\bar{U}-c)\phi_{b,y}]B(x_1) , \tag{3.23}$$

and as $\eta \to 0$,

$$p_{b1} \sim P_{b1}(x_1) + O(\eta^2) , \quad \text{with} \quad P_{b1}(x_1) = \mathrm{i}\alpha_0^{-1}\bar{U}_j'\tilde{b}_j B(x_1) . \tag{3.24}$$

Likewise, for the oblique mode and the forced difference mode, we have

$$p_{a1} = \mathrm{i}\alpha^{-1}\cos^2\theta[\bar{U}'\phi_a - (\bar{U}-c)\phi_{a,y}]A(x_1) , \tag{3.25}$$

$$p_{d1} = \mathrm{i}\alpha_d^{-1}\cos^2\theta_d[\bar{U}'v_{d1} - (\bar{U}-c)v_{d1,y}] , \tag{3.26}$$

where we define

$$\theta = \tan^{-1}(\beta/\alpha) , \quad \theta_d = \tan^{-1}(\beta/\alpha_d) . \tag{3.27}$$

As $\eta \to 0$,

$$p_{a1} \sim P_{a1}(x_1) + O(\eta^2), p_{d1} \sim P_{d1}(x_1) + O(\eta^2),$$

with

$$P_{a1}(x_1) = \mathrm{i}\alpha^{-1}\bar{U}_j'\cos^2\theta\, b_j A(x_1) , \quad P_{d1}(x_1) = \mathrm{i}\alpha_d^{-1}\bar{U}_j'\cos^2\theta_d\, D_j(x_1) . \tag{3.28}$$

The leading-order streamwise and spanwise velocities of the oblique mode are given, as in Wu (1992), by

$$u_{a1} = -\frac{1}{\mathrm{i}\alpha}\left(\left[\frac{\bar{U}'}{\bar{U}-c}\phi_a - \phi_{a,y}\right]\sin^2\theta + \phi_{a,y}\right)A(x_1) , \tag{3.29}$$

$$w_{a1} = -\frac{\sin\theta\cos\theta}{\mathrm{i}\alpha}\left(\frac{\bar{U}'}{\bar{U}-c}\phi_a - \phi_{a,y}\right)A(x_1) , \tag{3.30}$$

and as $\eta \to 0$,

$$u_{a1} \sim -\frac{\sin^2\theta\, b_j A(x_1)}{\mathrm{i}\alpha\eta} + \dots , \quad w_{a1} \sim -\frac{\sin\theta\cos\theta\, b_j A(x_1)}{\mathrm{i}\alpha\eta} + \dots . \tag{3.31}$$

Likewise, the corresponding quantities for the forced difference mode are found to have the asymptotic behaviour

$$u_{d1} \sim -\frac{\sin^2\theta_d\, D_j(x_1)}{\mathrm{i}\alpha_d\eta} + \dots , \quad w_{d1} \sim \frac{\sin\theta_d\cos\theta_d\, D_j(x_1)}{\mathrm{i}\alpha_d\eta} + \dots . \tag{3.32}$$

For both the oblique eigenmode and the forced difference mode, the streamwise and spanwise velocities exhibit a pole type of singularity. Other terms in the expansions (3.1) are not needed in the present study.

4. Inner expansion

Within the jth critical layer, the appropriate local transverse coordinate is

$$Y = (y - y_j)/\mu \, . \tag{4.1}$$

The expansion takes the following form:

$$\left.\begin{aligned}
p &= \delta[P_{a1}(x_1) + \mu P_{a2}(x_1) + \mu^2 P_{a3}(x_1, Y) + \ldots]\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon[P_{b1}(x_1) + \ldots]\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\
&\quad + \epsilon\delta\mu^{-3}[P_{d1}(x_1) + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + c.c. + \ldots \, . \\
u &= \delta[\mu^{-1}U_{a0}(x_1, Y) + U_{a1} + \mu U_{a2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon[U_{b1} + \mu U_{b2} + \ldots]\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\
&\quad + \epsilon\delta\mu^{-3}[\mu^{-1}U_{d0} + U_{d1} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \epsilon\delta\mu^{-3}[U_{s1} + \mu U_{s2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} \\
&\quad + \delta^2\mu^{-3}[U_{aa} + \ldots]\mathrm{e}^{2\mathrm{i}(\alpha\zeta - \beta z)} + c.c. + \delta^2\mu^{-3}U_M + \ldots \, , \\
v &= \delta[V_{a0}(x_1) + \mu V_{a1}(x_1, Y) + \mu^2 V_{a2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} \\
&\quad + \epsilon[V_{b0}(x_1) + \mu V_{b1}(x_1, Y) + \mu^2 V_{b2} + \ldots]\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\
&\quad + \epsilon\delta\mu^{-3}[V_{d0}(x_1) + \mu V_{d1}(x_1, Y) + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} \\
&\quad + \epsilon\delta\mu^{-2}[V_{s1}(x_1, Y) + \mu V_{s2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} + c.c. + \ldots \, , \\
w &= \delta[\mu^{-1}W_{a0}(x_1, Y) + W_{a1} + \mu W_{a2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha\zeta - \beta z)} + \epsilon[W_{b1} + \mu W_{b2} + \ldots]\mathrm{e}^{\mathrm{i}\alpha_0\zeta} \\
&\quad + \epsilon\delta\mu^{-3}[\mu^{-1}W_{d0} + W_{d1} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_d\zeta + \beta z)} + \epsilon\delta\mu^{-3}[W_{s1} + \mu W_{s2} + \ldots]\mathrm{e}^{\mathrm{i}(\alpha_s\zeta - \beta z)} \\
&\quad + \delta^2\mu^{-3}[W_{aa} + \ldots]\mathrm{e}^{2\mathrm{i}(\alpha\zeta - \beta z)} + c.c. + \delta^2\mu^{-3}W_M + \ldots \, ,
\end{aligned}\right\} \tag{4.2}$$

with P_{a1}, P_{b1}, and P_{d1} given by (3.24) and (3.28), and

$$V_{a0}(x_1) = A_j(x_1) \equiv b_j A(x_1) \, , \quad V_{b0}(x_1) = B_j(x_1) \equiv \tilde{b}_j B(x_1) \, , \tag{4.3}$$

$$V_{d0}(x_1) = D_j(x_1) \, . \tag{4.4}$$

Strictly speaking, the expansions (4.2) also contain terms in $\log \mu$, but these play a purely passive role and have been suppressed.

The leading-order streamwise and spanwise velocity components of the oblique mode satisfy

$$\hat{\mathrm{L}}_\alpha U_{a0} + U_j' V_{a0}(x_1) = -\mathrm{i}\alpha P_{a1}(x_1) \, , \quad \hat{\mathrm{L}}_\alpha W_{a0} = \mathrm{i}\beta P_{a1}(x_1) \, , \tag{4.5}$$

where we define

$$\hat{\mathrm{L}}_\alpha = \frac{\partial}{\partial x_1} + \mathrm{i}\alpha(\bar{U}_j' Y + S) - \lambda \frac{\partial^2}{\partial Y^2} \, . \tag{4.6}$$

It follows that

$$U_{a0} = -\bar{U}_j' \sin^2\theta \; Q_a^{(0)} \, , \quad W_{a0} = -\bar{U}_j' \sin\theta \cos\theta \; Q_a^{(0)} \, , \tag{4.7}$$

where

$$Q_a^{(n)}(x_1, Y) = \int_0^\infty \xi^n A_j(x_1 - \xi) \mathrm{e}^{-\Lambda_j \xi^3 - \mathrm{i}\alpha(\bar{U}_j' Y + S)\xi} \, \mathrm{d}\xi \, , \quad \Lambda_j = \tfrac{1}{3}\lambda\alpha^2 \bar{U}_j'^2 \, . \tag{4.8}$$

The leading-order velocity components of the difference mode, U_{d0} and W_{d0}, are driven by the pressure P_{d1}. They are much larger than the locally forced solutions U_{d1} and W_{d1}. We find that

$$\hat{\mathrm{L}}_{\alpha_d} U_{d0} + U_j' V_{d0}(x_1) = -\mathrm{i}\alpha_d P_{d1}(x_1) \, , \quad \hat{\mathrm{L}}_{\alpha_d} W_{d0} = -\mathrm{i}\beta P_{d1}(x_1) \, , \tag{4.9}$$

and thus

$$U_{d0} = -\bar{U}_j' \sin^2\theta_d \; Q_d^{(0)} \, , \quad W_{d0} = \bar{U}_j' \sin\theta_d \cos\theta_d \; Q_d^{(0)} \, , \tag{4.10}$$

where

$$Q_d^{(n)}(x_1, Y) = \int_0^\infty \xi^n D_j(x_1 - \xi) \mathrm{e}^{-\Lambda_j v_d^2 \xi^3 - \mathrm{i}\alpha_d(\bar{U}_j' Y + S)\xi} \, \mathrm{d}\xi \, . \tag{4.11}$$

Here for later convenience, we define

$$\sigma = \frac{\alpha}{\alpha_0}\,, \quad \sigma_d = \frac{\alpha_d}{\alpha_0}\,, \quad \sigma_s = \frac{\alpha_s}{\alpha_0}\,; \quad v_0 = \frac{\alpha_0}{\alpha}\,, \quad v_d = \frac{\alpha_d}{\alpha}\,, \quad v_s = \frac{\alpha_s}{\alpha}\,. \tag{4.12}$$

The second-order difference-mode terms are obtained from

$$c^{-1}U_{d0,x_1} + \mathrm{i}\alpha_d U_{d1} + V_{d1,Y} + \mathrm{i}\beta W_{d1} = 0\,, \tag{4.13}$$

$$\begin{aligned}\hat{\mathrm{L}}_{\alpha_d} U_{d1} + U_j' V_{d1} &= -V_{b0}U^*_{a0,Y} - U_j'' Y V_{d0} + \ldots \\ &= \mathrm{i}\alpha \bar{U}_j'^2 \sin^2\theta B_j\, Q_a^{(1)*} - \bar{U}_j'' Y D_j + \ldots\,,\end{aligned} \tag{4.14}$$

$$\hat{\mathrm{L}}_{\alpha_d} W_{d1} = -V_{b0} W^*_{a0,Y} + \ldots = \mathrm{i}\beta^{-1}\alpha^2 \bar{U}_j'^2 \sin^2\theta B_j\, Q_a^{(1)*} + \ldots\,, \tag{4.15}$$

where the first term on the right-hand sides arises from the interaction between the planar and the oblique modes. It follows that

$$\hat{\mathrm{L}}_{\alpha_d} V_{d1,YY} = \mathrm{i}\alpha_0\alpha^2 \bar{U}_j'^3 \sin^2\theta B_j(x_1) Q_a^{(2)*} + \mathrm{i}\alpha_d \bar{U}_j'' D_j(x_1) + \ldots\,, \tag{4.16}$$

which may be solved by means of a Fourier transform with respect to Y to give

$$\begin{aligned} V_{d1,YY} = {} & \mathrm{i}\alpha_0\alpha^2 \bar{U}_j'^3 \sin^2\theta \int_0^\infty\!\!\int_0^\infty \zeta^2 \mathrm{e}^{-\Lambda_j \sigma_d^{-1}\zeta^3 - \Lambda_j v_d^{-1}(v_d\eta-\zeta)^3} \mathrm{e}^{-\mathrm{i}\alpha(\bar{U}_j' Y + S)(v_d\eta-\zeta)} \\ & \times B_j(x_1-\eta) A_j^*(x_1-\eta-\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ & + \mathrm{i}\alpha_d \bar{U}_j'' \int_0^\infty \mathrm{e}^{-\Lambda_j v_d^2\eta^3 - \mathrm{i}\alpha_d(\bar{U}_j' Y + S)\eta} D_j(x_1-\eta)\,\mathrm{d}\eta + \ldots\,. \end{aligned} \tag{4.17}$$

Matching $V_{d1,Y}$ with the outer solution gives

$$C_j^+ - C_j^- - \pi\,\mathrm{i}\frac{\bar{U}_j''}{|\bar{U}_j'|} D_j(x_1) = N_j F_j(x_1)\,, \tag{4.18}$$

where the inhomogenous forcing term on the right-hand side is specified by

$$N_j = 2\pi\,\mathrm{i}\alpha_0^{-2}\alpha^2\alpha_d^2 \sin^2\theta \bar{U}_j' |\bar{U}_j'| \tilde{b}_j b_j^*\,, \tag{4.19}$$

$$F_j(x_1) = \int_0^\infty \eta^2 \mathrm{e}^{-\Lambda_j \sigma_d^2 \eta^3} B(x_1 - \sigma\eta) A^*(x_1-\eta)\,\mathrm{d}\eta\,. \tag{4.20}$$

Later, we shall also require the solution for the spanwise velocity, which is obtained from (4.15) as

$$\begin{aligned} W_{d1} = {} & \mathrm{i}\beta^{-1}\alpha^2 \bar{U}_j'^2 \sin^2\theta \int_0^\infty\!\!\int_0^\infty \zeta \mathrm{e}^{-\Lambda_j \sigma_d^{-1}\zeta^3 - \Lambda_j v_d^{-1}(v_d\eta-\zeta)^3} \mathrm{e}^{-\mathrm{i}\alpha(\bar{U}_j' Y + S)(v_d\eta-\zeta)} \\ & \times B_j(x_1-\eta) A_j^*(x_1-\eta-\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta + \ldots\,. \end{aligned} \tag{4.21}$$

The continuity equation (4.13) then gives the streamwise velocity:

$$U_{d1} = -(\mathrm{i}\alpha_d)^{-1}(V_{d1,Y} + \mathrm{i}\beta W_{d1} + c^{-1}U_{d0,x_1})\,. \tag{4.22}$$

Likewise, the leading-order sum mode satisfies

$$\mathrm{i}\alpha_s U_{s1} + V_{s1,Y} - \mathrm{i}\beta W_{s1} = 0\,, \tag{4.23}$$

$$\hat{\mathrm{L}}_{\alpha_s} U_{s1} + U_j' V_{s1} = -V_{b0} U_{a0,Y} = -\mathrm{i}\alpha \bar{U}_j'^2 \sin^2\theta B_j\, Q_a^{(1)}\,, \tag{4.24}$$

$$\hat{\mathrm{L}}_{\alpha_s} W_{s1} = -V_{b0} W_{a0,Y} = -\mathrm{i}\beta^{-1}\alpha^2 \bar{U}_j'^2 \sin^2\theta B_j\, Q_a^{(1)}\,, \tag{4.25}$$

from which we obtain

$$\hat{\mathrm{L}}_{\alpha_s} V_{s1,YY} = \mathrm{i}\alpha_0\alpha^2 \bar{U}_j'^3 \sin^2\theta\, Q_a^{(2)}\,. \tag{4.26}$$

Equation (4.26) can be solved by the same method as for V_{d1} to give

$$V_{s1,YY} = \mathrm{i}\alpha_0\alpha^2\bar{U}_j'^3\sin^2\theta\int_0^\infty\int_0^\infty \zeta^2 \mathrm{e}^{-A_j\sigma_s^{-1}\zeta^3 - A_j\nu_s^{-1}(\nu_s\eta+\zeta)^3}\mathrm{e}^{-\mathrm{i}\alpha(\bar{U}_j'Y+S)(\nu_s\eta+\zeta)} \times B_j(x_1-\eta)A_j(x_1-\eta-\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta\,, \qquad (4.27)$$

while (4.25) gives

$$W_{s1} = -\mathrm{i}\beta^{-1}\alpha^2\bar{U}_j'^2\sin^2\theta\int_0^\infty\int_0^\infty \zeta \mathrm{e}^{-A_j\sigma_s^{-1}\zeta^3 - A_j\nu_s^{-1}(\nu_s\eta+\zeta)^3}\mathrm{e}^{-\mathrm{i}\alpha(\bar{U}_j'Y+S)(\nu_s\eta+\zeta)} \times B_j(x_1-\eta)A_j(x_1-\eta-\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta\,. \qquad (4.28)$$

It follows from (4.23) that

$$U_{s1} = -(\mathrm{i}\alpha_s)^{-1}(V_{s1,Y} - \mathrm{i}\beta W_{s1})\,. \qquad (4.29)$$

We note that, unlike the forced difference mode, the leading-order sum mode decays (algebraically) as $Y \to \pm\infty$, and hence does not affect the flow outside the critical layers. It can be shown (although we do not present the calculation) that the same is true of the second-order sum mode (U_{s2}, V_{s2}, W_{s2}). This is why the outer solution contains the difference mode at $O(\epsilon\delta\mu^{-3})$, while the sum mode is smaller, of $O(\epsilon\delta\mu^{-1})$.

The mean-flow distortions U_M and W_M satisfy

$$\left[\frac{\partial}{\partial x_1} - \lambda\frac{\partial^2}{\partial Y^2}\right] U_M = -[V_{a0}U_{a0,Y}^* + \text{c.c.}] = \mathrm{i}\alpha\bar{U}_j'^2\sin^2\theta A_j Q_a^{(2)*} + \text{c.c.}\,,$$
$$\left[\frac{\partial}{\partial x_1} - \lambda\frac{\partial^2}{\partial Y^2}\right] W_M = -[V_{a0}W_{a0,Y}^* + \text{c.c.}] = \mathrm{i}\beta^{-1}\alpha^2\bar{U}_j'^2\sin^2\theta A_j Q_a^{(2)*} + \text{c.c.}\,, \qquad (4.30)$$

and are given by

$$U_M = \mathrm{i}\alpha\bar{U}_j'^2\sin^2\theta\int_0^\infty\int_0^\infty \zeta \mathrm{e}^{-A_j(\zeta^3+3\zeta^2\eta)}\mathrm{e}^{\mathrm{i}\alpha(\bar{U}_j'Y+S)\zeta}A_j(x_1-\eta)A_j^*(x_1-\eta-\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta + \text{c.c.}\,,$$
$$W_M = (\alpha/\beta)U_M\,. \qquad (4.31)$$

For the streamwise and spanwise velocities of the oblique harmonic, $e^{2\mathrm{i}(\alpha\xi-\beta z)}$, it suffices to note (from the continuity equation) that

$$W_{aa} = (\alpha/\beta)U_{aa}\,.$$

It turns out that this harmonic does not contribute to the nonlinear terms in the final amplitude equations.

We turn now to the oblique-mode corrections V_{a1} and V_{a2}. They are found to satisfy

$$\hat{\mathrm{L}}_\alpha V_{a1,YY} = \mathrm{i}\alpha\bar{U}_j'' A_j(x_1)\,, \qquad (4.32)$$

$$\hat{\mathrm{L}}_\alpha V_{a2,YY} = [\hat{\mathrm{L}}_1(\alpha)V_{a1} + \hat{\mathrm{L}}_2(\alpha,\beta)A_j] + N_{a2}\,, \qquad (4.33)$$

where we have defined

$$\hat{\mathrm{L}}_1(\alpha) = -\left\{\mathrm{i}\alpha\left[\tfrac{1}{2}\bar{U}_j''Y^2 + \lambda c x_1\frac{\partial\bar{U}_j}{\partial x_3}\right] + c^{-1}\bar{U}_j'Y\frac{\partial}{\partial x_1}\right\}\frac{\partial^2}{\partial Y^2} + \mathrm{i}\alpha\bar{U}_j''\,, \qquad (4.34)$$

$$\hat{\mathrm{L}}_2(\alpha,\beta) = (\alpha^2+\beta^2)\hat{\mathrm{L}}_\alpha + \left[\mathrm{i}\alpha\bar{U}_j'''Y + c^{-1}\bar{U}_j''\frac{\partial}{\partial x_1}\right]\,. \qquad (4.35)$$

The final term of (4.33) represents the interaction between the planar wave and the forced difference mode (α_d, β, c), and is given by

$$N_{a2} = \mathrm{i} V_{b0}\left(\alpha U_{d0}^{*} - \beta W_{d0}^{*}\right)_{YY} = \mathrm{i}\alpha_0\alpha_d^2 \bar{U}_j'^3 \sin^2\theta_d B_j Q_d^{(2)*} + \dots \ . \tag{4.36}$$

Solving (4.32) and matching $V_{a1,Y}$ and V_{a1} with the outer expansion, we obtain the jump conditions

$$a_j^+ - a_j^- = \pi\,\mathrm{i} p_j b_j \,\mathrm{sgn}(\bar{U}_j') \ , \quad d_j^+ - d_j^- = -\pi\,\mathrm{i} r_j b_j \,\mathrm{sgn}(\bar{U}_j') \ . \tag{4.37}$$

After solving for $V_{a2,YY}$, we obtain

$$\begin{aligned} c_j^+ - c_j^- &\equiv \mathrm{FP}\left[V_{a2,Y}(Y=\infty) - V_{a2,Y}(Y=-\infty)\right] \\ &= \pi\,\mathrm{i}\,\mathrm{sgn}(\bar{U}_j')(a_j^+ r_j + p_j d_j^+ + b_j s_j) \\ &\quad +2\pi\,\mathrm{i}\alpha_0^{-2}\alpha^2\alpha_d^2\bar{U}_j'|\bar{U}_j'|\sin^2\theta_d \int_0^\infty \xi^2 \mathrm{e}^{-\Lambda_j\sigma_d^2\xi^3} B_j(x_1-\sigma_d\xi) D_j^{*}(x_1-\xi)\,\mathrm{d}\xi \ . \end{aligned} \tag{4.38}$$

Finally we consider the planar mode. For the purpose of deriving the amplitude equation, it suffices to seek the solution for V_{b1} and V_{b2} only. We find that V_{b1} satisfies the equation

$$\hat{\mathrm{L}}_{\alpha_0} V_{b1,YY} = \mathrm{i}\alpha_0 \bar{U}_j'' B_j + \mathrm{i}\alpha_0 R_{b1,Y} \ , \tag{4.39}$$

where

$$R_{b1} = 2\,\mathrm{i}\alpha_0 U_{d0} U_{a0} + V_{d0} U_{a0,Y} + V_{a0} U_{d0,Y} \ . \tag{4.40}$$

Solving (4.39) and matching $V_{b1,Y}$ and V_{b1} with the outer solution gives the jumps

$$\tilde{a}_j^+ - \tilde{a}_j^- = \pi\,\mathrm{i} p_j \tilde{b}_j \,\mathrm{sgn}(\bar{U}_j') \ , \quad \tilde{d}_j^+ - \tilde{d}_j^- = -\pi\,\mathrm{i} \tilde{r}_j \tilde{b}_j \,\mathrm{sgn}(\bar{U}_j') \ , \tag{4.41}$$

The forcing term $\mathrm{i}\alpha_0 R_{b1,Y}$ in (4.39), which represents the interaction of the oblique mode with the leading-order difference mode, makes no contribution to these jumps, unlike the corresponding terms in the oblique-mode equations (4.33). This accounts for the different scaling for the amplitudes of the planar mode and the oblique mode, as was noted in §2.

It remains to determine

$$\tilde{c}_j^+ - \tilde{c}_j^- \equiv \mathrm{FP}\left[V_{b2,Y}(Y=\infty) - V_{b2,Y}(Y=-\infty)\right] \ . \tag{4.42}$$

This can be obtained by solving for V_{b2}, which is found to satisfy

$$\hat{\mathrm{L}}_{\alpha_0} V_{b2,YY} = \hat{\mathrm{L}}_1(\alpha_0) V_{b1} + \hat{\mathrm{L}}_2(\alpha_0, 0) B_j + \mathrm{i}\alpha_0 R_{b2,Y} \ , \tag{4.43}$$

where

$$\begin{aligned} R_{b2} = {} & 2\,\mathrm{i}\alpha_0 U_{s1} U_{a0}^{*} + V_{s1} U_{a0,Y}^{*} + V_{s1,Y} U_{ao}^{*} + V_{a0}^{*} U_{s1,Y} \\ & +2\,\mathrm{i}\alpha_0 U_{d1} U_{a0} + V_{d1} U_{a0,Y} + V_{d1,Y} U_{ao} + V_{a0} U_{d1,Y} + V_{b0} U_{M,Y} + \dots \ . \end{aligned} \tag{4.44}$$

Let $V_{b2}^{(l)}$ and $V_{b2}^{(n)}$ denote the solutions driven by $[\hat{\mathrm{L}}_1 V_{b1} + \hat{\mathrm{L}}_2 B_j]$ and by $(\mathrm{i}\alpha_0 R_{b2,Y})$ respectively. We find that

$$\mathrm{FP}\left[V_{b2,Y}^{(l)}(Y=\infty) - V_{b2,Y}^{(l)}(Y=-\infty)\right] = \pi\,\mathrm{i}\,\mathrm{sgn}(\bar{U}_j')(\tilde{a}_j^+ \tilde{r}_j + p_j \tilde{d}_j^+ + \tilde{b}_j \tilde{s}_j) \ , \tag{4.45}$$

where the definitions for $\tilde{r}_j$ and $\tilde{s}_j$ are stated below (3.14).

In calculating the jump in $V^{(n)}_{b2,Y}$, it is helpful to make the substitution identified by Wu (1995), namely,

$$V^{(n)}_{b2,YY} = V^{(r)}_{b2,YY} + V^{(s)}_{b2,YY} , \tag{4.46}$$

where

$$V^{(s)}_{b2,YY} = -\frac{\mathrm{i}\alpha_0}{\bar{U}'_j}[U_{a0,YY}U_{d1} + U^*_{a0,YY}U_{s1}] , \tag{4.47}$$

so that $V^{(r)}_{b2,YY}$ satisfies

$$\hat{\mathrm{L}}_{\alpha_0}V^{(r)}_{b2,YY} = N^{(r)}_{b2} \equiv \mathrm{i}\alpha_0 R_{b2,Y} - \hat{\mathrm{L}}_{\alpha_0}V^{(s)}_{b2,YY}. \tag{4.48}$$

We note that

$$\hat{\mathrm{L}}_{\alpha_0}V^{(s)}_{b2,YY} = -\frac{\mathrm{i}\alpha_0}{\bar{U}'_j}\Big[U_{a0,YY}(\hat{\mathrm{L}}_{\alpha_d}U_{d1}) + U_{d1}(\hat{\mathrm{L}}_{\alpha}U_{a0,YY}) - 2\lambda U_{a0,YYY}U_{d1,Y}$$
$$+ U^*_{a0,YY}(\hat{\mathrm{L}}_{\alpha_d}U_{s1}) + U_{s1}(\hat{\mathrm{L}}_{\alpha}U_{a0,YY})^* - 2\lambda U^*_{a0,YYY}U_{s1,Y}\Big] , \tag{4.49}$$

and that after substitution from (4.14) and (4.24), the right-hand side can be written as

$$-\frac{\mathrm{i}\alpha_0}{\bar{U}'_j}\Big[U_{a0,YY}(-U'_jV_{d1} - V_{b0}U^*_{a0,Y} + \ldots) + U_{d1}(-2\,\mathrm{i}\alpha\bar{U}'_jU_{a0,Y}) - 2\lambda U_{a0,YYY}U_{d1,Y}$$
$$+ U^*_{a0,YY}(-U'_jV_{s1} - V_{b0}U_{a0,Y}) + U_{s1}(2\,\mathrm{i}\alpha\bar{U}'_jU^*_{a0,Y}) - 2\lambda U^*_{a0,YYY}U_{s1,Y}\Big] . \tag{4.50}$$

It follows that

$$\begin{aligned} N_{b2} = & -2\alpha_0\beta U^*_{a0,Y}W_{s1} - \frac{2\beta\alpha_0^2}{\alpha_s}U^*_{a0}W_{s1,Y} - \mathrm{i}\frac{\alpha_0\alpha_d}{\alpha_s}U^*_{a0}V_{s1,YY} \\ & + 2\alpha_0\beta U_{a0,Y}W_{d1} + \frac{2\beta\alpha_0^2}{\alpha_d}U_{a0}W_{d1,Y} - \mathrm{i}\frac{\alpha_0\alpha_s}{\alpha_d}U_{a0}V_{d1,YY} \\ & - \frac{\alpha_0}{\alpha_d}A_j(V_{d1,YYY} + \mathrm{i}\beta W_{d1,YY}) \\ & + \frac{2\lambda\alpha_0}{\alpha_s\bar{U}'_j}U^*_{a0,YYY}(V_{s1,YY} - \mathrm{i}\beta W_{s1,Y}) + \frac{2\lambda\alpha_0}{\alpha_d\bar{U}'_j}U_{a0,YYY}(V_{d1,YY} + \mathrm{i}\beta W_{d1,Y}) \\ & - \frac{\mathrm{i}\alpha_0}{\bar{U}'_j}B_jU_{a0,YY}U^*_{a0,Y} - \frac{\mathrm{i}\alpha_0}{\bar{U}'_j}B_jU^*_{a0,YY}U_{a0,Y} + \mathrm{i}\alpha_0B_j\bar{U}_{M,YY} + \ldots . \end{aligned} \tag{4.51}$$

By means of Fourier transform with respect to Y, it is now straightforward, though lengthy, to solve for $V^{(r)}_{b2,YY}$ and so find the jump in $V^{(r)}_{b2,Y}$. The jump in $V^{(s)}_{b2,Y}$ can be obtained directly from (4.47) by integration by parts. After a tedious calculation, we have

$$\begin{aligned} \tilde{c}^+_j - \tilde{c}^-_j = & \pi\,\mathrm{i}\,\mathrm{sgn}(\bar{U}'_j)(a^+_j\tilde{r}_j + p_j\tilde{d}^+_j + b_j\tilde{s}_j) \\ & + 2\pi\alpha^3|U'_j|^3\sin^2\theta\int_0^\infty\!\!\int_0^\infty K_b(\xi,\eta\,|\,\Lambda_j)A_j(x_1-\xi)B_j(x_1-\xi-\eta)A^*_j(x_1-v_s\xi-v_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ & + 2\pi\alpha^3|\bar{U}'_j|^3\sin^2\theta\int_0^\infty\!\!\int_0^\infty K_c(\xi,\eta\,|\,\Lambda_j)B_j(x_1-\xi)A_j(x_1-\xi-\eta)A^*_j(x_1-v_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta . \end{aligned} \tag{4.52}$$

The kernels K_b and K_c are rather complicated, and are given in Appendix A. But in the inviscid limit ($\lambda = 0$), they revert to a very simple form, namely

$$K_b(\xi,\eta \mid 0) = v_0^3 \cos^2\theta \; \xi(\xi+\eta)(v_0\xi + v_d\eta) , \tag{4.53}$$

$$K_c(\xi,\eta \mid 0) = v_0^4 \cos^2\theta \; \xi^3 . \tag{4.54}$$

5. Evolution equations for the amplitudes

5.1. *Case I: the planar wave interacts with a single oblique mode*

The jumps (4.37) and (4.38) are now substituted into (3.15) to give the amplitude-evolution equation for a single oblique mode :

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \int_0^\infty \sum_j \Upsilon^{(j)} \xi^2 \mathrm{e}^{-A_j \sigma_d^2 \xi^3} B(x_1 - \sigma_d \xi) D_j{}^*(x_1 - \xi)\,\mathrm{d}\xi , \tag{5.1}$$

where the sum is over all critical layers. The D_j are specified by (3.19)–(3.22) and (4.18), while the coefficients κ_a and $\Upsilon^{(j)}$ are given by

$$\kappa_a = -\,\mathrm{i}\alpha S \left[1 + \frac{2\alpha^2 J_1}{c c_g(\alpha)}\right] , \quad \Upsilon^{(j)} = -2\pi\alpha_0^{-2}\alpha^3\alpha_d^2 \sin^2\theta_d b_j \tilde{b}_j \bar{U}_j' |\bar{U}_j'| / c_g(\alpha) , \tag{5.2}$$

where

$$c_g(\alpha) = J_2 - \frac{2\alpha^2}{c} J_1 + \sum_j \left\{ 2\pi\,\mathrm{i} a_j^+ b_j \frac{\bar{U}_j''}{\bar{U}_j' |\bar{U}_j'|} + \pi\,\mathrm{i} b_j^2 \frac{\bar{U}_j'''\bar{U}_j' - \bar{U}_j''^2}{|\bar{U}_j'|^3} + \pi^2 b_j^2 \frac{\bar{U}_j''^2}{\bar{U}_j'^3} \right\} . \tag{5.3}$$

Likewise, substitution of (4.41) and (4.52) into (3.17) gives the amplitude equation for the planar mode, namely

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B + \int_0^\infty\!\!\int_0^\infty \sum_j \Upsilon_s^{(j)} K_b(\xi,\eta \mid A_j) A(x_1-\xi) B(x_1-\xi-\eta) A^*(x_1 - v_s\xi - v_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \int_0^\infty\!\!\int_0^\infty \sum_j \Upsilon_s^{(j)} K_c(\xi,\eta \mid A_j) B(x_1-\xi) A(x_1-\xi-\eta) A^*(x_1 - v_s\xi - \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta , \tag{5.4}$$

where

$$\kappa_b = -\,\mathrm{i}\alpha_0 S \left[1 + \frac{2\alpha_0^2 \tilde{J}_1}{c\tilde{c}_g(\alpha_0)}\right] , \quad \Upsilon_s^{(j)} = 2\pi\,\mathrm{i}\alpha_0\alpha^3 \sin^2\theta |b_j|^2 \tilde{b}_j^2 |\bar{U}_j'|^3 / \tilde{c}_g(\alpha_0) , \tag{5.5}$$

and

$$\tilde{c}_g(\alpha_0) = \tilde{J}_2 - \frac{2\alpha_0^2}{c}\tilde{J}_1 + \sum_j \left\{ 2\pi\,\mathrm{i}\tilde{a}_j^+ \tilde{b}_j \frac{\bar{U}_j''}{\bar{U}_j' |\bar{U}_j'|} + \pi\,\mathrm{i}\tilde{b}_j^2 \frac{\bar{U}_j'''\bar{U}_j' - \bar{U}_j''^2}{|\bar{U}_j'|^3} + \pi^2 \tilde{b}_j^2 \frac{\bar{U}_j''^2}{\bar{U}_j'^3} \right\} . \tag{5.6}$$

In order to match with the upstream linear stage, the amplitudes A and B should satisfy the 'initial' conditions (see e.g. Goldstein & Leib 1989):

$$A \to A_0 \mathrm{e}^{\kappa_a x_1} , \quad B \to B_0 \mathrm{e}^{\kappa_b x_1} \quad \text{as} \quad x_1 \to -\infty , \tag{5.7}$$

where the complex numbers A_0 and B_0 characterize the initial amplitudes of the oblique and the planar modes respectively.

5.2. *Case II: the planar wave interacts with a pair of oblique modes*

As we have emphasized, the phase-locked mechanism is essentially a binary modal interaction, between a planar mode and a *single* oblique mode. Nevertheless, it can be extended to the case where the planar wave interacts with a pair of oblique waves at equal and opposite angles to the streamwise direction. The disturbance then takes the form

$$\epsilon B(x_1)\mathrm{e}^{\mathrm{i}\alpha_0\zeta} + \delta A(x_1)[\mathrm{e}^{\mathrm{i}(\alpha\zeta+\beta z)} + \mathrm{e}^{\mathrm{i}(\alpha\zeta-\beta z)}] + \text{c.c.} + \ldots \, . \tag{5.8}$$

Here for simplicity, we have assumed that the two oblique modes have the same amplitude. In principle, it is straightforward to allow for unequal amplitudes, but the algebra is considerably more complicated. For the equal-amplitude case, the analysis is largely the same as in the previous sections, and will not be given in detail. The major difference is that the self-interaction between the oblique modes produce an additional spanwise-dependent mean-flow distortion and a harmonic component proportion to $\exp(2\,\mathrm{i}\alpha\zeta)$. These in turn interact, at the cubic level, with the leading-order oblique modes, to contribute an extra nonlinear term to the amplitude equation for the oblique modes. This term has been calculated by Goldstein & Choi (1989) in the inviscid limit and by Wu *et al.* (1993) with the inclusion of viscosity. The amplitude equation (5.1) is thus modified to

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \int_0^\infty \sum_j \Upsilon^{(j)}\xi^2 \mathrm{e}^{-A_j\sigma_d^2\xi^3} B(x_1-\sigma_d\xi)D_j^*(x_1-\xi)\,\mathrm{d}\xi$$
$$+\int_0^\infty\int_0^\infty \sum_j \Upsilon_a^{(j)} K_a(\xi,\eta\,|\,A_j)A(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\, , \tag{5.9}$$

where

$$\Upsilon_a^{(j)} = 4\pi\,\mathrm{i}\alpha^4\sin^2\theta b_j^2|b_j|^2|\bar{U}_j'|^3/c_g(\alpha)\, , \tag{5.10}$$

and the kernel K_a is given in Appendix B. The governing equation for the planar mode is

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B + \int_0^\infty\int_0^\infty \sum_j \Upsilon_p^{(j)} K_b(\xi,\eta\,|\,A_j)A(x_1-\xi)B(x_1-\xi-\eta)A^*(x_1-v_s\xi-v_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\int_0^\infty\int_0^\infty \sum_j \Upsilon_p^{(j)} K_c(\xi,\eta\,|\,A_j)B(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-v_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\, , \tag{5.11}$$

which differs from (5.4) only in that

$$\Upsilon_p^{(j)} = 2\Upsilon_s^{(j)} = 4\pi\,\mathrm{i}\alpha_0\alpha^3\sin^2\theta|b_j|^2\tilde{b}_j^2|\bar{U}_j'|^3/\tilde{c}_g(\alpha_0)\, . \tag{5.12}$$

This is because each of the two oblique modes makes an equal contribution to the amplitude equation. In what follows, we shall concentrate on (5.9) and (5.11); the equations for a single oblique mode can be recovered as a special case, with the $\Upsilon_a^{(j)}$ set to zero and the $\Upsilon_p^{(j)}$ replaced by half their value.

5.3. *Solvability condition for the forced difference mode*

We recall that the D_j in (5.1) or (5.9) are obtained as

$$D_j(x_1) = v_{d1}(x_1, y_j)\, , \tag{5.13}$$

where v_{d1} is a solution of the Rayleigh equation (3.19) for the forced difference mode,

$$\frac{\partial^2 v_{d1}}{\partial y^2} - \left(\alpha_d^2 + \beta^2 + \frac{\bar{U}''}{\bar{U}-c}\right) v_{d1} = 0 \tag{5.14}$$

with the jump conditions

$$[v_{d1}]_{y_j^-}^{y_j^+} = 0 , \quad \left[v_{d1,y}\right]_{y_j^-}^{y_j^+} - \mathrm{i}\pi \frac{\bar{U}_j''}{|\bar{U}_j'|} v_{d1}(x_1, y_j) = N_j F_j(x_1) \tag{5.15}$$

and boundary conditions

$$v_{d1} \to 0 \quad \text{as } y \to \pm\infty . \tag{5.16}$$

Here N_j and F_j are specified by (4.19)–(4.20). In the foregoing analysis, it has been tacitly assumed that the problem specified by (5.14)–(5.16) has a solution. This is indeed the case if (α_d, β, c) is not an eigenmode of the system, i.e. there is no non-trivial solution of the Rayleigh equation

$$\phi_e'' - \left(\alpha_d^2 + \beta^2 + \frac{\bar{U}''}{\bar{U} - c}\right) \phi_e = 0 , \tag{5.17}$$

with

$$[\phi_e]_{y_j^-}^{y_j^+} = 0 , \quad \left[\phi_e'\right]_{y_j^-}^{y_j^+} - \mathrm{i}\pi \frac{\bar{U}_j''}{|\bar{U}_j'|} \phi_e(y_j) = 0 , \quad \phi_e \to 0 \quad \text{as } y \to \pm\infty . \tag{5.18}$$

Suppose, on the other hand, that the homogeneous problem (5.17)–(5.18) does have a solution, with

$$\phi_e \sim f_j[1 + p_j \eta \log|\eta|] + e_j^{\pm} \eta + O(\eta^2 \log|\eta|) \quad \text{as} \quad \eta \equiv y - y_j \to \pm 0 . \tag{5.19}$$

By multiplying (5.14) by ϕ_e and integrating over y, we obtain the solvability condition for ϕ_d, namely

$$\sum_j \left[f_j(C_j^+ - C_j^-) - (e_j^+ - e_j^-)D_j\right] = 0 , \quad \text{i.e.} \quad \sum_j f_j N_j F_j(x_1) = 0 . \tag{5.20}$$

In this case, v_{d1} plus an arbitary multiple of ϕ_e satisfies (5.14) and (5.15). In order for v_{d1} and the amplitude of the oblique mode(s) to be precisely defined, we should also impose a normalization condition on v_{d1}, say

$$\int_{-\infty}^{\infty} \phi_e(y) v_{d1}(x_1, y)\, \mathrm{d}y = 0 . \tag{5.21}$$

A case in point arises when the oblique mode is the subharmonic of the planar mode, that is, when

$$\alpha = \tfrac{1}{2}\alpha_0 , \quad \beta = \sqrt{3}\alpha , \quad \theta = 60^\circ , \tag{5.22}$$

since then $\alpha_d = \alpha$, and (5.17)–(5.18) is satisfied by $\phi_e = \phi_a$. The solvability condition (5.20) thus becomes

$$\sum_j b_j N_j F_j(x_1) = 0 , \tag{5.23}$$

which can be interpreted as cancellation of the quadratic subharmonic resonance. For asymmetric flows, or for *varicose* modes in symmetric shear flows, (5.23) cannot in general be satisfied, and it must be assumed that $\theta \neq 60^\circ$. The subharmonic resonance may then be of particular importance because it requires an $O(\mu^4)$ magnitude of the planar mode (see e.g. Wu 1992, 1995), much smaller than that for the phase-locked interaction to occur. For *sinuous* modes in a symmetric shear flow, the solvability condition (5.23) is satisfied when $\theta = 60^\circ$ and consequently subharmonic resonance is inactive among such modes. On the other hand, there may be some value of

θ for which the difference mode coincides with a *varicose* eigenmode, as will be demonstrated in §6 for a particular wake profile. In that case, the appropriate solvability condition cannot be satisfied. It is then necessary to treat the (varicose) difference mode on the same footing as the two sinuous modes, and a *three-mode* 'resonant-triad' interaction ensues. This is explored in detail by Wu (1996), who shows that such a 'mixed-mode' triad can arise in a wide variety of wake profiles. Like the subharmonic triad, it comes into play when the two-dimensional mode has an magnitude of $O(\mu^4)$, and so provides a possible mechanism for the selective amplification of a narrow band of oblique modes. However, the non-resonant phase-locked interaction can lead to the enhanced growth of other oblique modes once the magnitude of the planar mode rises to $O(\mu^{7/2})$. Since it is not restricted to $\theta = 60°$, it may help to explain why a broad band of three-dimensional disturbances can be observed to develop rapidly.

5.4. *Simplified form of the amplitude equations*

In general, the evolution of $A(x_1)$ and $B(x_1)$ is governed by (5.9) and (5.11) together with (5.14) and (5.15); these equations must be solved simultaneously to determine the development of $A(x_1)$, $B(x_1)$ and the $D_j(x_1)$. However, considerable simplification is obtained if all the critical layers have the same value of Λ_j, as defined in (4.8), so that the F_j are all equal. This happens, for example,

(*a*) if there is only one critical layer (as will be the case if the basic flow profile is monotonic), or

(*b*) if the basic flow profile is symmetric, with two critical layers (as for a plane wake or jet), or

(*c*) in the inviscid limit, $\lambda = 0$.

From now on, except in §7.1, we shall always make this simplifying assumption, and denote the common values of Λ_j and F_j by Λ and F. We can then write

$$v_{d1} = F(x_1)\phi_d(y) , \tag{5.24}$$

where ϕ_d satisfies

$$\phi_d'' - \left(\alpha_d^2 + \beta^2 + \frac{\bar{U}''}{\bar{U} - c}\right)\phi_d = 0 , \tag{5.25}$$

$$[\phi_d]_{y_j^-}^{y_j^+} = 0 , \quad [\phi_d']_{y_j^-}^{y_j^+} - \mathrm{i}\pi\frac{\bar{U}_j''}{|\bar{U}_j'|}\phi_d(y_j) = N_j , \quad \phi_d \to 0 \quad \text{as } y \to \pm\infty. \tag{5.26}$$

Note that $D_j = F(x_1)\phi_d(y_j)$. After substituting it into (5.9), the amplitude equation for the oblique modes becomes

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \Upsilon\int_0^\infty\int_0^\infty K(\xi,\eta\,|\,\Lambda)B(x_1 - \sigma_d\xi)B^*(x_1 - \xi - \sigma\eta)A(x_1 - \xi - \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\Upsilon_a\int_0^\infty\int_0^\infty K_a(\xi,\eta\,|\,\Lambda)A(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta , \tag{5.27}$$

where κ_a is defined in (5.2), and

$$\left.\begin{aligned} K(\xi,\eta\,|\,\Lambda) = \xi^2\eta^2\mathrm{e}^{-\Lambda\sigma_d^2(\xi^3+\eta^3)} , \quad \Upsilon_a = 4\pi\,\mathrm{i}\alpha^4\sin^2\theta\left\{\textstyle\sum_j b_j^2|b_j|^2|\bar{U}_j'|^3\right\}/c_g(\alpha), \\ \Upsilon = -2\pi\alpha_0^{-2}\alpha^3\alpha_d^2\sin^2\theta_d\left\{\textstyle\sum_j b_j\bar{b}_j\phi_d^*(y_j)\bar{U}_j'|\bar{U}_j'|\right\}/c_g(\alpha). \end{aligned}\right\} \tag{5.28}$$

Note that equation (5.25) with the jump condition (5.26) can be solved independently of $A(x_1)$ and $B(x_1)$ to determine $\phi_d(y_j)$ and hence Υ.

The governing equation (5.11) for the planar mode also simplifies, to

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B + \Upsilon_p \int_0^\infty \int_0^\infty K_b(\xi,\eta\,|\,\Lambda)A(x_1-\xi)B(x_1-\xi-\eta)A^*(x_1-v_s\xi-v_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\Upsilon_p \int_0^\infty \int_0^\infty K_c(\xi,\eta\,|\,\Lambda)B(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-v_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \qquad (5.29)$$

where κ_b is specified by (5.5), and

$$\Upsilon_p = 4\pi\,\mathrm{i}\alpha_0\alpha^3\sin^2\theta\left\{\sum_j |b_j|^2\tilde{b}_j^2|\bar{U}_j'|^3\right\}/\tilde{c}_g(\alpha_0)\,. \qquad (5.30)$$

As before, the equations for a single oblique mode are recovered by dropping the last term of (5.27) and replacing the coefficient Υ_p in (5.29) by $\Upsilon_s = \Upsilon_p/2$.

We end this section by making the rescaling

$$x_1 = (\bar{x}+\bar{x}_0)/\kappa_{br}\,, \qquad (5.31)$$

$$A(x_1) = A_0\mathrm{e}^{\kappa\bar{x}_0+\mathrm{i}\sigma\kappa_{bi}x_1}\bar{A}(\bar{x})\,, \quad B(x_1) = B_0\mathrm{e}^{\bar{x}_0+\mathrm{i}\kappa_{bi}x_1}\bar{B}(\bar{x})\,, \qquad (5.32)$$

where κ_{br} and κ_{bi} are the real and imaginary parts of κ_b, and

$$\bar{x}_0 = \ln\left[\kappa_{br}^{7/2}|B_0|^{-1}|\Upsilon|^{-1/2}\right]\,, \quad \kappa = (\kappa_a - \mathrm{i}\sigma\kappa_{bi})/\kappa_{br}\,. \qquad (5.33)$$

After the rescaling, the amplitude equations (5.27) and (5.29) and upstream conditions (5.7) become

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}\bar{x}} = \kappa\bar{A} + \mathrm{e}^{\mathrm{i}\phi}\int_0^\infty\int_0^\infty K(\xi,\eta\,|\,\bar{\Lambda})\bar{B}(\bar{x}-\sigma_d\xi)\bar{B}^*(\bar{x}-\xi-\sigma\eta)\bar{A}(\bar{x}-\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\chi_0\bar{\Upsilon}_a\int_0^\infty\int_0^\infty K_a(\xi,\eta\,|\,\bar{\Lambda})\bar{A}(\bar{x}-\xi)\bar{A}(\bar{x}-\xi-\eta)\bar{A}^*(\bar{x}-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \qquad (5.34)$$

$$\frac{\mathrm{d}\bar{B}}{\mathrm{d}\bar{x}} = \bar{B} + \chi_0\bar{\Upsilon}_p\int_0^\infty\int_0^\infty K_b(\xi,\eta\,|\,\bar{\Lambda})\bar{A}(\bar{x}-\xi)\bar{B}(\bar{x}-\xi-\eta)\bar{A}^*(\bar{x}-v_s\xi-v_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\chi_0\bar{\Upsilon}_p\int_0^\infty\int_0^\infty K_c(\xi,\eta\,|\,\bar{\Lambda})\bar{B}(\bar{x}-\xi)\bar{A}(\bar{x}-\xi-\eta)\bar{A}^*(\bar{x}-v_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \qquad (5.35)$$

$$\bar{A}\to\mathrm{e}^{\kappa\bar{x}}\,, \quad \bar{B}\to\mathrm{e}^{\bar{x}} \quad \text{as} \quad \bar{x}\to-\infty\,, \qquad (5.36)$$

where

$$\mathrm{e}^{\mathrm{i}\phi} = \Upsilon/|\Upsilon|\,, \quad \bar{\Upsilon}_a = \Upsilon_a/|\Upsilon_p|\,, \quad \bar{\Upsilon}_p = \Upsilon_p/|\Upsilon_p|\,, \quad \bar{\lambda} = \lambda/(\kappa_{br})^3\,, \qquad (5.37)$$

and

$$\bar{\Lambda} = \tfrac{1}{3}\bar{\lambda}\alpha^2\bar{U}_1^2\,, \quad \chi_0 = \left\{\kappa_{br}^{7\gamma-6}|\Upsilon_p|/|\Upsilon|^\gamma\right\}\frac{|A_0|^2}{|B_0|^{2\gamma}}\,, \quad \gamma = \kappa_{ar}/\kappa_{br}\,. \qquad (5.38)$$

For a particular set of modes, the solutions of (5.34)–(5.36) depend only on two (real) parameters: $\bar{\lambda}$, the rescaled viscous (Haberman) parameter, and χ_0, a measure of the initial amplitude of the oblique modes relative to that of the planar mode. The evolution differs from that of a resonant triad in that it does not depend on the initial phase difference between the planar and the oblique modes. This feature might be used to distinguish the two mechanisms experimentally (see §8).

6. Application to a plane wake

From now on, the analysis will be specialized to the case of a symmetric plane wake. In such a flow (and indeed in any symmetric shear flow) subharmonic resonance is possible among varicose modes (v antisymmetric), but not among sinuous modes (v symmetric). Nevertheless, in the experiments of Corke *et al.* (1992), sinuous subharmonics excited in conjunction with a two-dimensional mode were observed to grow much faster than predicted by linear theory. Mallier (1995) suggests that the varicose modes might grow rapidly as a result of subharmonic resonance, and that once sufficiently large, they could enhance the growth of the sinuous modes. However, since varicose modes have much smaller linear growth rates than their sinuous counterparts, it seems doubtful whether such a scenario would occur in practice unless the varicose modes were preferentially excited by some means. Here we seek an alternative explanation in terms of a phase-locked interaction involving sinuous modes.

The basic-flow profile is chosen to be

$$\bar{U} = U_0 - \operatorname{sech}^2 y , \tag{6.1}$$

where U_0 is the (scaled) free-stream velocity. For neutral sinuous modes (Drazin & Howard 1966),

$$\alpha_0 = \left(\alpha^2 + \beta^2\right)^{1/2} = 2 , \quad c = U_0 - \tfrac{2}{3} , \quad \phi_a = \phi_b = \operatorname{sech}^2 y . \tag{6.2}$$

The two critical levels are located at $y_{1,2} = \frac{1}{2}\ln(2+\sqrt{3})$, which coincide with the inflexion points of $\bar{U}$. We thus have

$$\bar{U}_1' = -\bar{U}_2' = \frac{4}{3\sqrt{3}} , \quad \bar{U}_1'' = \bar{U}_2'' = 0 , \quad \bar{U}_1''' = -\bar{U}_2''' = -\frac{16}{3\sqrt{3}} ; \tag{6.3}$$

$$J_1 = \tilde{J}_1 = \tfrac{4}{3} , \quad J_2 = \tilde{J}_2 = 8\left[\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}) + 2\right] , \quad b_1 = \tilde{b}_1 = b_2 = \tilde{b}_2 = \tfrac{2}{3} . \tag{6.4}$$

It follows from (5.3), (5.6) and (5.5) that

$$c_g(\alpha) = \tilde{c}_g(\alpha) = 8\left[\frac{1}{\sqrt{3}}\ln(2+\sqrt{3}) + 2 - \frac{\alpha^2}{(3U_0-2)}\right] - \frac{8\pi\,\mathrm{i}}{\sqrt{3}} . \tag{6.5}$$

Substitution of the above values into (5.2), (5.5), (5.10) and (5.12) yields the values of κ_a, κ_b, Υ_a and Υ_p.

The determination of the remaining coefficient Υ requires the solution of the difference mode problem (5.14)–(5.25), which in general has to be obtained numerically. However, in the special case of an oblique subharmonic, that is, $\alpha = \alpha_d = 1$, $\beta = \sqrt{3}$, $\theta = 60°$, the solution of (5.25) which vanishes as $y \to \pm\infty$ can be expressed in the analytic form

$$\phi_d = \phi_f(y) + K \operatorname{sech}^2 y , \tag{6.6}$$

where

$$\phi_f = \begin{cases} \gamma_1 \operatorname{sech}^2 y , & y > y_1 ; \\ \gamma_2 \operatorname{sech}^2 y [\frac{1}{2}\sinh 4y + 4\sinh 2y + 6y] , & -y_1 < y < y_1 ; \\ \gamma_3 \operatorname{sech}^2 y , & y < -y_1 . \end{cases} \tag{6.7}$$

The constants γ_k are obtained from the jump conditions (5.26). Noting that

$$N_1 = -N_2 = -\frac{8\pi\,\mathrm{i}}{81} \tag{6.8}$$

we find that

$$\gamma_2 = \frac{\pi \mathrm{i}}{343}\,, \quad \gamma_1 = -\gamma_3 = \frac{\pi \mathrm{i}}{81}\left[2\sqrt{3} + \log(2+\sqrt{3})\right]\,. \tag{6.9}$$

The use of the normalization condition (5.21) determines $K = 0$. It follows that ϕ_d is *varicose* in character (i.e. odd in y). The coefficient Υ in (5.27) is found to be

$$\Upsilon = -\frac{16\pi^2}{6561}\left[2\sqrt{3} + \log(2+\sqrt{3})\right]/c_g(1)\,. \tag{6.10}$$

For a general value of θ, we may write

$$\Upsilon = 2\pi^2\alpha_0^4 \bar{U}_1'^4 |\bar{b}_1|^2 b_1 \left[\frac{\sin^4 2\theta\, \sin^6\frac{\theta}{2}\, \bar{\phi}(y_1\,|\,\theta)}{c_g(\alpha_0\cos\theta)}\right]\,, \tag{6.11}$$

where $\bar{\phi}(y\,|\,\theta)$ is the solution of

$$\bar{\phi}_{yy} - \left(\alpha_d^2 + \beta^2 + \frac{\bar{U}''}{\bar{U}-c}\right)\bar{\phi} = \left(16\sin^2\frac{\theta}{2} - 6\,\mathrm{sech}^2\,y\right)\bar{\phi}\,, \tag{6.12}$$

$$\bar{\phi}(0) = 0\,, \quad \left[\bar{\phi}_y\right]_{y_1^-}^{y_1^+} = 1\,, \quad \bar{\phi} \to 0 \quad \text{as} \quad y \to \infty\,. \tag{6.13}$$

The values of Υ depend, through c_g, on the choice of U_0. Figure 2 shows Υ plotted against θ for $U_0 = 11/3$, which is the value used in the numerical solutions to be presented in §7. We see that there is a critical value of θ at which Υ is infinite, namely

$$\theta_{crit} = \cos^{-1}(0.875) \approx 29^\circ\,. \tag{6.14}$$

For this value of θ (which does not depend on the choice of U_0), the difference mode coincides with a *varicose* eigenmode, since then

$$\alpha_d^2 + \beta^2 = 1\,, \tag{6.15}$$

and (5.17)–(5.18) has the solution

$$\phi_e = \mathrm{sech}\,y \tanh y\,. \tag{6.16}$$

It is at once apparent that the solvability condition (5.20) is not satisfied since $f_1 = -f_2$, $N_1 = -N_2$, and $F_1 = F_2$. This is the case of 'mixed-mode' resonance mentioned at the end of §5.3, and requires a separate analysis (Wu 1996). Here, however, we shall restrict attention to the non-resonant interaction. In fact, our numerical results will be confined to the case of $\theta = 60^\circ$, which corresponds to the situation investigated experimentally by Corke *et al.* (1992).

7. Study of the amplitude equations

7.1. *Strongly viscous limit: $\lambda \to \infty$*

If the initial disturbances are very small, nonlinear effects may become important further downstream and closer to the neutral station, so that the growth-rate parameter $\mu \ll R^{-1/3}$. In that case, non-equilibrium effects in the critical layer(s) are small compared to those of viscosity. The appropriate amplitude equations may be formally obtained from the foregoing analysis by taking the limit $\lambda \to \infty$, where λ is the 'Haberman parameter' defined by (2.16). We return, for this subsection only, to the full equations (5.9)–(5.15). A procedure similar to that in Wu *et al.* (1993) shows

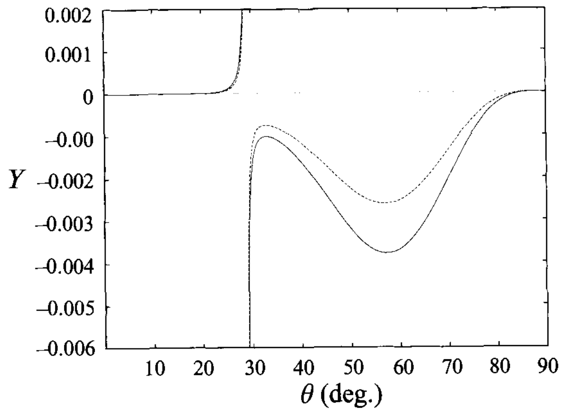

FIGURE 2. The coefficient Υ as a function of θ, for the plane-wake profile (6.1) with $U_0 = 11/3$. —, Re Υ, - - -, Im Υ.

that (5.9)–(5.26) reduce to

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \lambda^{-2}\check{\Upsilon}|B|^2 A + \lambda^{-4/3}\check{\Upsilon}_a A \int_0^\infty |A(x_1-\xi)|^2\,\mathrm{d}\xi + \dots\,, \tag{7.1}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B + \lambda^{-5/3}\check{\Upsilon}_p |A|^2 B + \dots\,, \tag{7.2}$$

where

$$\check{\Upsilon} = \frac{\alpha}{\alpha_d \alpha_0^2}\sum_j \frac{\Upsilon^{(j)}\check{\phi}_d^*(y_j)}{(\alpha\bar{U}_j')^2}\,, \quad \check{\Upsilon}_a = (\tfrac{16}{3})^{2/3}\sin^2\theta\cos 2\theta\,\Gamma(\tfrac{1}{3})\sum_j \frac{\Upsilon_a^{(j)}}{(\alpha\bar{U}_j')^{8/3}}\,, \tag{7.3}$$

$$\check{\Upsilon}_p = \left[\sum_j \frac{\Upsilon_p^{(j)}}{(\alpha\bar{U}_j')^{10/3}}\right]\int_0^\infty\int_0^\infty \Big[K_b(\xi,\eta\,|\,1) + K_c(\xi,\eta\,|\,1)\Big]\,\mathrm{d}\xi\,\mathrm{d}\eta\,, \tag{7.4}$$

and $\check{\phi}_d(y)$ is the solution of (5.25)–(5.26) with N_j replaced by $N_j/(\bar{U}_j')^2$.

For the case where the planar mode interacts with a single oblique mode, the integral term in (7.1) is absent, and $\check{\Upsilon}_p$ is replaced by $\check{\Upsilon}_s \equiv \check{\Upsilon}_p/2$. The parameter λ can be rescaled from the equations by writing $A = \lambda^{5/6}\check{A} + \dots$, $B = \lambda\check{B} + \dots$, to obtain

$$\frac{\mathrm{d}\check{A}}{\mathrm{d}x_1} = \kappa_a\check{A} + \check{\Upsilon}|\check{B}|^2\check{A}\,, \tag{7.5}$$

$$\frac{\mathrm{d}\check{B}}{\mathrm{d}x_1} = \kappa_b\check{B} + \check{\Upsilon}_s|\check{A}|^2\check{B}\,. \tag{7.6}$$

If $\mathrm{Re}\check{\Upsilon} < 0$ and $\mathrm{Re}\check{\Upsilon}_s < 0$, equations (7.5) and (7.6) admit an equilibrium solution of the form

$$\check{A} = \check{A}_0\mathrm{Re}^{\mathrm{i}\omega_a x_1}\,, \quad \check{B} = \check{B}_0\mathrm{Re}^{\mathrm{i}\omega_b x_1}\,,$$

where

$$|\check{A}_0|^2 = -\mathrm{Re}\kappa_b/\mathrm{Re}\check{\Upsilon}_s\,, \quad \omega_a = (\mathrm{Re}\check{\Upsilon}\mathrm{Im}\kappa_a - \mathrm{Re}\kappa\mathrm{Im}\check{\Upsilon})/\mathrm{Re}\check{\Upsilon}\,,$$

$$|\check{B}_0|^2 = -\mathrm{Re}\kappa_a/\mathrm{Re}\check{\Upsilon}\,, \quad \omega_b = (\mathrm{Re}\check{\Upsilon}_s\mathrm{Im}\kappa_b - \mathrm{Re}\kappa_b\mathrm{Im}\check{\Upsilon}_s)/\mathrm{Re}\check{\Upsilon}_s\,.$$

When a pair of oblique modes is present, the appropriate rescaling is $A = \lambda^{2/3}\check{A} + \dots$,

$B = \lambda \breve{B} + \dots$, which gives

$$\frac{\mathrm{d}\breve{A}}{\mathrm{d}x_1} = \kappa_a \breve{A} + \breve{\Upsilon}|\breve{B}|^2 \breve{A} + \breve{\Upsilon}_a \breve{A} \int_0^\infty |\breve{A}(x_1 - \xi)|^2 \,\mathrm{d}\xi \,, \tag{7.7}$$

$$\frac{\mathrm{d}\breve{B}}{\mathrm{d}x_1} = \kappa_b \breve{B} \,. \tag{7.8}$$

In this case, the feedback term cannot be retained in the leading-order equation. A discussion similar to that of Wu *et al.* (1993) indicates that the amplitude equations (7.5)–(7.6) or (7.7)–(7.8) break down when $\lambda = O(R^{1/2})$ because the non-parallelism is then a leading-order effect. Nevertheless, the appropriate amplitude equations for this new régime can be obtained by replacing the linear terms in (7.5) and (7.6) by $(\kappa_a + \rho_a x_1)$ and $(\kappa_b + \rho_b x_1)$ respectively, where $\rho_{a,b}$ are some suitable constants; the nonlinear terms remain intact.

7.2. *'Secondary-instability' stage, and effect of initial amplitude*

The coupled amplitude equations (5.34) and (5.35) are formally derived by assuming $\delta \sim O(\epsilon^{6/7})$. If $\delta \ll O(\epsilon^{6/7})$, i.e. $|\chi_0| \ll 1$; then they reduce to

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}\bar{x}} = \kappa \bar{A} + \mathrm{e}^{\mathrm{i}\bar{\phi}} \int_0^\infty \!\! \int_0^\infty K(\xi, \eta \mid \bar{\Lambda}) \bar{B}(\bar{x} - \sigma_d \xi) \bar{B}^*(\bar{x} - \xi - \sigma \eta) \bar{A}(\bar{x} - \xi - \eta) \,\mathrm{d}\xi \,\mathrm{d}\eta \,, \tag{7.9}$$

$$\frac{\mathrm{d}\bar{B}}{\mathrm{d}\bar{x}} = \bar{B} \,. \tag{7.10}$$

These equations show that the oblique mode(s) are influenced by the mutual interaction even if it is infinitesimal, while the planar mode evolves according to linear theory, unaffected by the oblique mode. We shall refer to this régime as the *secondary-instability* stage, since it turns out that the planar wave is unstable to three-dimensional modes which share the same phase velocity. However, we should point out that the present secondary-instability analysis differs from that of Herbert (1988), where it is assumed that the planar wave is in (quasi-)equilibrium, or can be treated as such in an *ad hoc* manner, so that an eigenvalue problem can be formulated using Floquet theory. In the present case, the non-equilibrium nature of the planar wave is properly taken into account, and the secondary instability is an *initial-value problem* as (7.9), (7.10) and (5.36) indicate.

As in Goldstein & Lee (1992), a solution of (7.9) and (7.10) subject to the initial condition (5.36) can be found in the form

$$\bar{B} = \mathrm{e}^{\bar{x}} \,, \quad \bar{A} = \mathrm{e}^{\kappa \bar{x}} \left[1 + \sum_{n=1}^{\infty} a_n \mathrm{e}^{2n\bar{x}} \right] , \tag{7.11}$$

with the a_n $(n = 1, 2, \dots)$ given by

$$a_n = \frac{\mathrm{e}^{\mathrm{i}n\bar{\phi}}}{2^{5n} n!} \left[\frac{\Gamma(p)\Gamma(q)}{\Gamma(n+p)\Gamma(n+q)} \right]^3 \prod_{m=0}^{n-1} I_m(p) I_m(q) \,, \tag{7.12}$$

where

$$I_n(p) = 4(n+p)^3 \int_0^\infty \xi^2 \mathrm{e}^{-\bar{\Lambda}_d \xi^3 - 2(n+p)\xi} \,\mathrm{d}\xi \,, \tag{7.13}$$

$$p = \tfrac{1}{2}(\sigma_d + \kappa + 1) \,, \quad q = \tfrac{1}{2}(\sigma + \kappa) \,, \quad \bar{\Lambda}_d = \bar{\Lambda}\sigma_d^2 = \tfrac{1}{3}\bar{\lambda} \left(\alpha_d \bar{U}_1' \right)^2 \,. \tag{7.14}$$

Note that $I_n \equiv 1$ for the inviscid case ($\bar{\Lambda}_d = 0$), and that for *any* value of $\bar{\Lambda}_d$,

$$I_n(p) \approx 1 \quad \text{for} \quad |n+p| \gg \left(\bar{\Lambda}_d\right)^{1/3} . \tag{7.15}$$

Application of Laplace's method (e.g. Bender & Orszag 1978) to the series in (7.11) gives, for $|\bar{\phi}| < \pi$,

$$\bar{A}(\bar{x}) \sim \bar{A}_\infty \exp\left[\bar{k}e^{2\bar{x}/7} + \tfrac{1}{7}\kappa\bar{x} - \tfrac{3}{7}\,\mathrm{i}\kappa\bar{\phi}\right] \quad \text{as} \quad \bar{x} \to \infty , \tag{7.16}$$

where

$$\bar{A}_\infty = \left[\frac{\Gamma(p)\Gamma(q)}{2\pi}\right]^3 \frac{32^{3\kappa/7}}{\sqrt{7}} \prod_{m=0}^{\infty} I_m(p)I_m(q) , \quad \bar{k} = \frac{7}{32^{\frac{1}{7}}}e^{\mathrm{i}\bar{\phi}/7} \tag{7.17}$$

For $\bar{\phi} = \pm\pi$, that is, for Υ real and negative, the corresponding large-$\bar{x}$ result is

$$\bar{A}(\bar{x}) \sim \bar{A}_\infty \left(\exp\left[\bar{k}e^{2\bar{x}/7} + \tfrac{1}{7}\kappa\bar{x} - \tfrac{3}{7}\,\mathrm{i}\kappa\bar{\phi}\right] + \exp\left[\bar{k}^*e^{2\bar{x}/7} + \tfrac{1}{7}\kappa\bar{x} + \tfrac{3}{7}\,\mathrm{i}\kappa\bar{\phi}\right]\right) . \tag{7.18}$$

This implies that for any value of $\bar{\phi}$, that is for any (non-zero) value of the coefficient Υ, the oblique mode grows *super-exponentially*. The planar mode acts as a 'catalyst' to enhance the growth of the oblique mode.

The amplitude of the forced difference mode is characterized by

$$\bar{F}(\bar{x}) = \int_0^\infty \bar{\eta}^2 e^{-\bar{\Lambda}_d\bar{\eta}^3} B(\bar{x} - \sigma\bar{\eta})\bar{A}^*(\bar{x} - \bar{\eta})\,\mathrm{d}\bar{\eta} . \tag{7.19}$$

Substitution of (7.11) into (7.19) yields a power-series solution for $\bar{F}(\bar{x})$. After a similar analysis as was used for (7.11), it is found that (for $|\bar{\phi}| < \pi$)

$$\bar{F}(\bar{x}) \sim \bar{F}_\infty \exp\left[\bar{k}^*e^{2\bar{x}/7} + \tfrac{1}{7}(\kappa^* + 1)\bar{x} + \tfrac{3}{7}(1 + \kappa^*)\,\mathrm{i}\bar{\phi}\right] \quad \text{as} \quad \bar{x} \to \infty, \tag{7.20}$$

where

$$\bar{F}_\infty = \frac{1}{4}\left[\frac{\Gamma(p^*)\Gamma(q^*)}{2\pi}\right]^3 \frac{32^{3(1+\kappa^*)/7}}{\sqrt{7}} \prod_{m=0}^{\infty} I_m(p^*)I_m(q^*) . \tag{7.21}$$

As a result of such rapid growth, the amplitude of oblique waves may eventually overtake that of the planar wave. Depending on the initial magnitude of the oblique modes as well as on the nature of the critical layers, several possibilities can occur.

(a) If the oblique modes are algebraically small initially, their magnitude quickly increases to $O(\epsilon^{6/7})$ so that the evolution soon enters the fully coupled régime. In this case the fully coupled equations (5.34) and (5.35) are uniformly valid in the two régimes.

(b) However, if the initial magnitude of the oblique modes is exponentially small and the critical layers are *regular*, then the two-dimensional mode can become nonlinear, that is, described by strongly nonlinear critical-layer equations such as those of Goldstein & Hultgren (1989), before the oblique modes can produce a feedback effect. Finally the disturbance may enter a fully coupled stage in which both the planar and the oblique modes evolve over an inviscid spatial scale. Because the three-dimensional modes have a smaller growth rate in the linear stage, such a scenario is likely to occur in experiments. We are currently investigating this issue.

(c) If the oblique modes are exponentially small initially, but the critical layers are *singular*, then the secondary-instability stage is followed by one in which the development of the planar mode is governed by an evolution equation of Hickernell (1984) type. Because the amplitude of the planar mode can either develop a finite-distance singularity or equilibrate (see e.g. Wu & Cowley 1995), the subsequent stage may be different from that for a regular critical layer.

7.3. *Fully coupled stage: a finite-distance singularity*

It appears that the solution to (5.34) and (5.35) can develop a finite-distance singularity, say at x_s, of the following form:

$$\bar{A} \sim \frac{a_0}{(\bar{x}_s - \bar{x})^{3+\mathrm{i}\psi_1}} , \quad \bar{B} \sim \frac{b_0}{(\bar{x}_s - \bar{x})^{7/2+\mathrm{i}\psi_2}} \quad \text{as} \quad \bar{x} \to \bar{x}_s , \tag{7.22}$$

where a_0, b_0 are complex in general, while ψ_1 and ψ_2 are real. Substitution of (7.22) into (5.34) and (5.35) yields

$$(3 + \mathrm{i}\psi_1) = \mathrm{e}^{\mathrm{i}\bar{\phi}} D_{11}|b_0|^2 + \chi_0 \bar{\Upsilon}_a D_{12}|a_0|^2 , \tag{7.23}$$

$$(\tfrac{7}{2} + \mathrm{i}\psi_2) = \chi_0 \bar{\Upsilon}_p \cos^2\theta (v_0^3 D_{21} + v_0^4 D_{22})|a_0|^2 , \tag{7.24}$$

where D_{11}, etc. are defined by the following convergent integrals:

$$\left.\begin{aligned}
D_{11}(\psi_1,\psi_2) &= \int_0^\infty\!\!\int_0^\infty \xi^2\eta^2(1+\sigma_d\xi)^{-(7/2+\mathrm{i}\psi_2)}(1+\sigma\eta+\xi)^{-(7/2-\mathrm{i}\psi_2)}(1+\xi+\eta)^{-(3+\mathrm{i}\psi_1)}\,\mathrm{d}\xi\,\mathrm{d}\eta , \\
D_{22}(\psi_1,\psi_2) &= \int_0^\infty\!\!\int_0^\infty \xi^3(1+\xi)^{-(7/2+\mathrm{i}\psi_2)}(1+\xi+\eta)^{-(3+\mathrm{i}\psi_1)}(1+v_s\xi+\eta)^{-(3-\mathrm{i}\psi_1)}\,\mathrm{d}\xi\,\mathrm{d}\eta , \\
D_{12}(\psi_1) &= \int_0^\infty\!\!\int_0^\infty K_a(\xi,\eta\,|\,0)[(1+\xi)(1+\xi+\eta)]^{-(3+\mathrm{i}\psi_1)}(1+2\xi+\eta)^{-(3-\mathrm{i}\psi_1)}\,\mathrm{d}\xi\,\mathrm{d}\eta , \\
D_{21}(\psi_1,\psi_2) &= \int_0^\infty\!\!\int_0^\infty \xi(\xi+\eta)(v_0\xi+v_d\eta)(1+\xi+\eta)^{-(7/2+\mathrm{i}\psi_2)} \\
&\qquad \times(1+\xi)^{-(3+\mathrm{i}\psi_1)}(1+v_s\xi+v_0\eta)^{-(3-\mathrm{i}\psi_1)}\,\mathrm{d}\xi\,\mathrm{d}\eta ,
\end{aligned}\right\} \tag{7.25}$$

and the kernel K_a is given by (B 4). Equations (7.23) and (7.24) determine ψ_1, ψ_2, $|a_0|$ and $|b_0|$, while x_s can be fixed in the same way as described in Wu (1992). Note that the singularity structure is independent of the viscosity parameter $\bar{\lambda}$. However, the distance at which the singularity occurs is delayed by viscosity, as our numerical results will show.

7.4. *Numerical study of the amplitude equations*

We now present numerical solutions of the (rescaled) amplitude equations (5.34) and (5.35). The finite-difference method employed is an Adams–Moulton (implicit) scheme with sixth-order accuracy. The kernels K, K_a, etc. are evaluated numerically by Simpson's rule. As in Wu *et al.* (1993) and Wu (1995), the integrals over the infinite domains (see (5.34), (5.35)) are approximated by those over large but finite domains, the sizes of which are determined by trial and error.

The coefficients that we shall use are those calculated analytically in §5 for sinuous subharmonics in the plane-wake profile (6.1). They depend on U_0, which in turn is related to the velocity deficit. This changes considerably from the near field to the fully developed similarity region (see e.g. Sato & Kuriki 1961; Sato & Saito 1978). In the following calculations, we choose $U_0 = 11/3$, which corresponds to a velocity deficit of 0.272, very close to that in the experiments of Corke *et al.* (1992). We also note that in most experiments, the velocity deficit in the far wake is about this size. The other two parameters are χ_0 and $\bar{\lambda}$, which represent the initial magnitude of the oblique modes and the effect of viscosity (or Reynolds number) respectively.

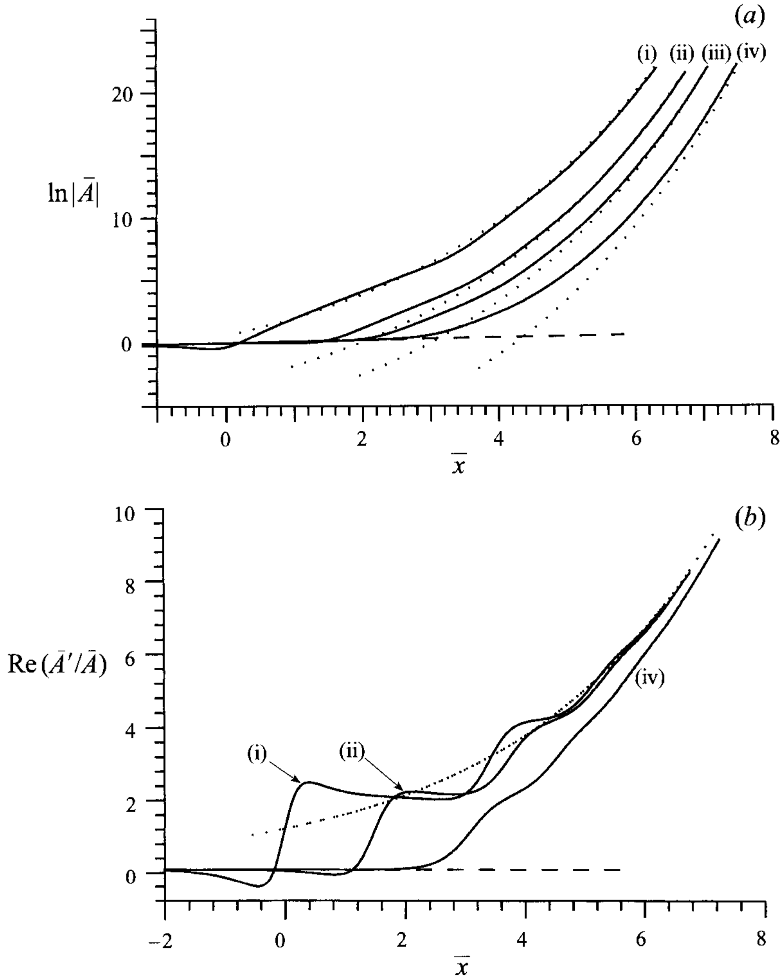

FIGURE 3(*a,b*). For caption see facing page.

We first study the amplitude equations governing the 'secondary-instability' stage, i.e. (7.9) and (7.10). (The results can also be obtained from the series solution (7.11)–(7.14), which provides a useful check on the numerical scheme.) The development of the oblique wave is shown in figure 3(*a*) for four different values of $\bar{\lambda}$. We find that the linear growth rate of the oblique mode is only about one-tenth that of the planar mode. However, its growth is soon significantly enhanced through phase-locked interaction with the planar mode once the latter has attained a certain magnitude. The phase-locked interaction finally leads to the super-exponential growth in the amplitude of the oblique mode. This implies that even if the oblique mode is very small initially, it may eventually overtake the planar mode. Viscosity has a stabilizing effect in that it postpones the action of the phase-locked interaction, but it cannot prevent the ultimate super-exponential growth. Figure 3(*b*) shows the instantaneous growth rate of the oblique mode, $\mathrm{Re}(\bar{A}'/\bar{A})$. It is interesting to note that in the inviscid and moderately viscous case, the growth rate exhibits a step-function-like behaviour. That is, once the phase-locked interaction comes into play, the growth rate 'jumps' to a higher value and remains more or less constant for some distance. Over this

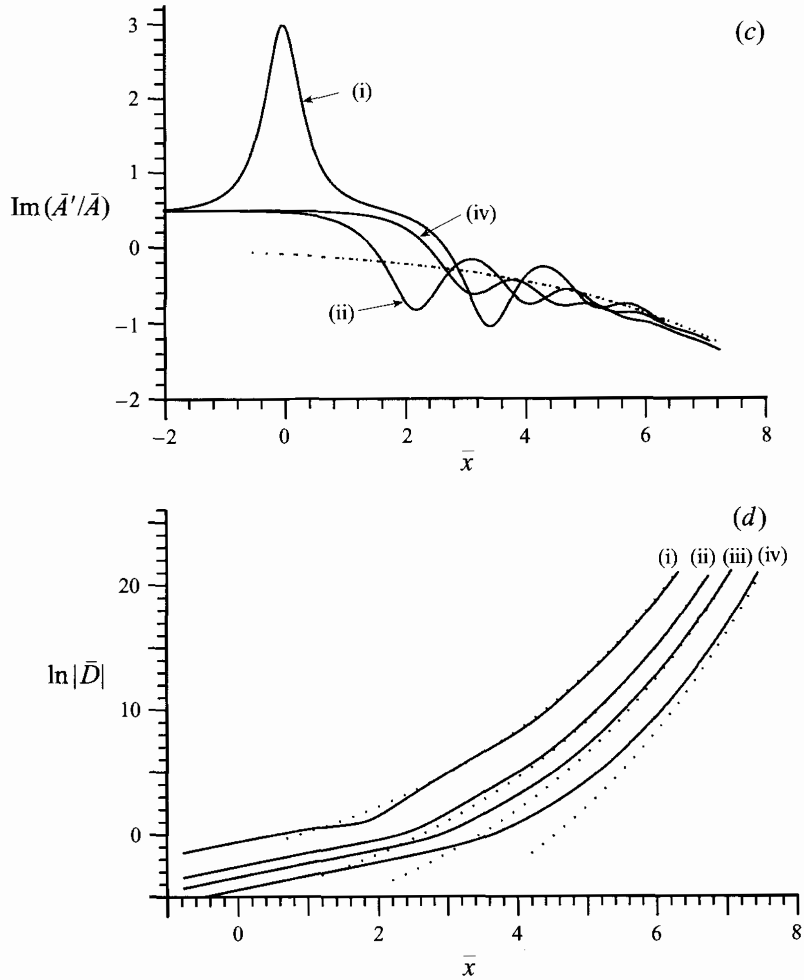

FIGURE 3. (*a*) Ln$|\bar{A}|$ *vs.* $\bar{x}$. The dotted lines represent the large-distance asymptotic behaviour (7.16) and the dashed line the exponential growth. (*b*) The instantaneous growth rate Re$(\bar{A}'/\bar{A})$ *vs.* $\bar{x}$. The dotted line represents the asymptotic result. (*c*) The wavenumber correction Im$(\bar{A}'/\bar{A})$ *vs.* $\bar{x}$. The dotted line represents the asymptotic result. (*d*) Ln$|\bar{D}|$ *vs.* $\bar{x}$. The dotted lines represent the large-distance asymptotic behaviour (7.20). (i) $\bar{\lambda} = 0$, (ii) $\bar{\lambda} = 5$, (iii) $\bar{\lambda} = 15$, and (iv) $\bar{\lambda} = 50$.

range, the amplitude grows almost exponentially but with growth rate about twenty times the linear value, or twice the growth rate of the planar mode. For the slightly viscous case, $\bar{\lambda} = 5.0$, it can be seen that the immediate effect of the phase-locked interaction is to cause a transient decay before enhancing the amplification. These findings are in agreement with the experiments of Corke *et al.* (1992), who observed the oblique waves to decay slightly in the initial stage of the interaction, and then evolve exponentially with a growth rate comparable to that of the planar mode. The step-function-like behaviour of the instantaneous growth rate may be the reason why Corke *et al.* were able to fit the nonlinear development of the oblique wave with an exponential function. As well as causing a super-exponential growth in the amplitude, the phase-locked interaction induces a wavenumber correction, Im$(\bar{A}'/\bar{A})$, as shown in figure 3(*c*). This may lead to an appreciable change in the effective wavenumber of the disturbance. Indeed, the simultaneous modulation of amplitude and wavenumber

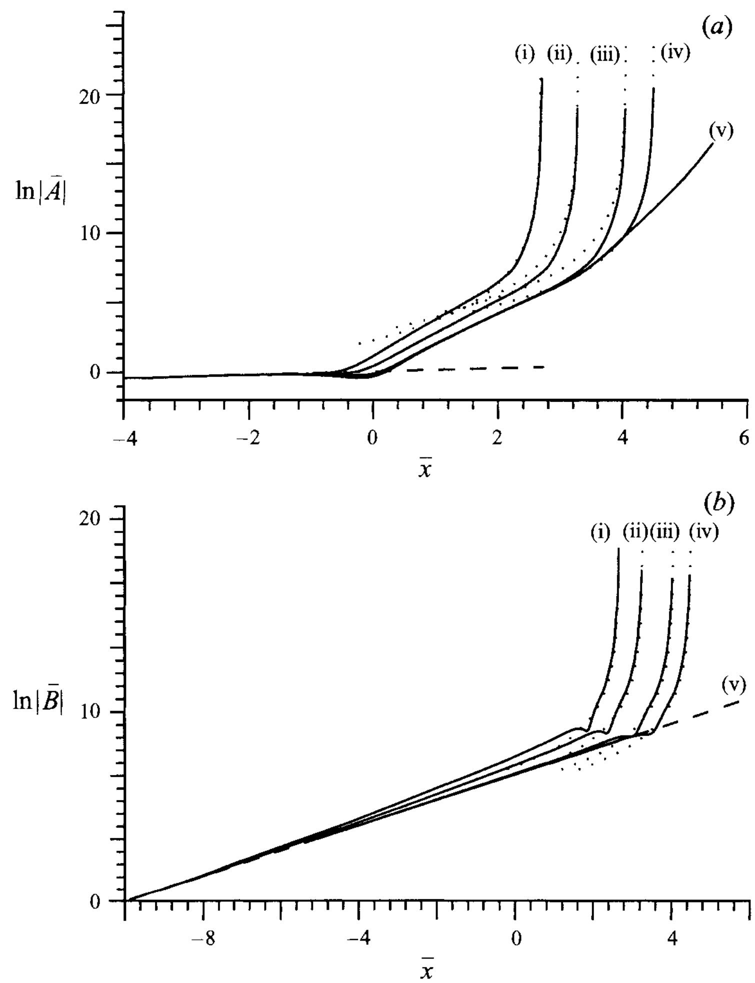

FIGURE 4. (*a*) Ln$|\bar{A}|$ *vs*. $\bar{x}$ and (*b*) ln$|\bar{B}|$ *vs*. $\bar{x}$ for $\bar{\lambda} = 0$: (i) $\chi_0 = 0.1$, (ii) $\chi_0 = 0.05$, (iii) $\chi_0 = 10^{-2}$, (iv) $\chi_0 = 10^{-3}$, (v) $\chi_0 = 0$. The dotted lines represent the local singular solution (7.22), and the dashed line the exponential growth.

is characteristic of transition in a plane wake and in a mixing layer (Miksad, *et al.* 1982; Miksad 1972).

Our analysis shows that the induced difference mode has a much larger magnitude than that of the sum mode. Moreover, a comparison between (7.16) and (7.20) indicates that the difference mode ultimately grows more rapidly than the primary oblique modes by a factor of $e^{\bar{x}/7}$. The development of the difference mode is depicted in figure 3(*d*).

Figures 3(*a*)–3(*d*) all show that as $\bar{x} \to +\infty$, the numerical solutions asymptote to the predicted super-exponential growth. Since the growth of the planar mode is still only exponential, this terminal behaviour could have been obtained from a secondary-instability calculation in which the planar mode was treated as being in equilibrium. However, such an *ad hoc* analysis could not capture the intriguing transient behaviour, which, as we have noted, is in broad agreement with experimental observations.

We now study the amplitude equations governing the fully coupled stage, which

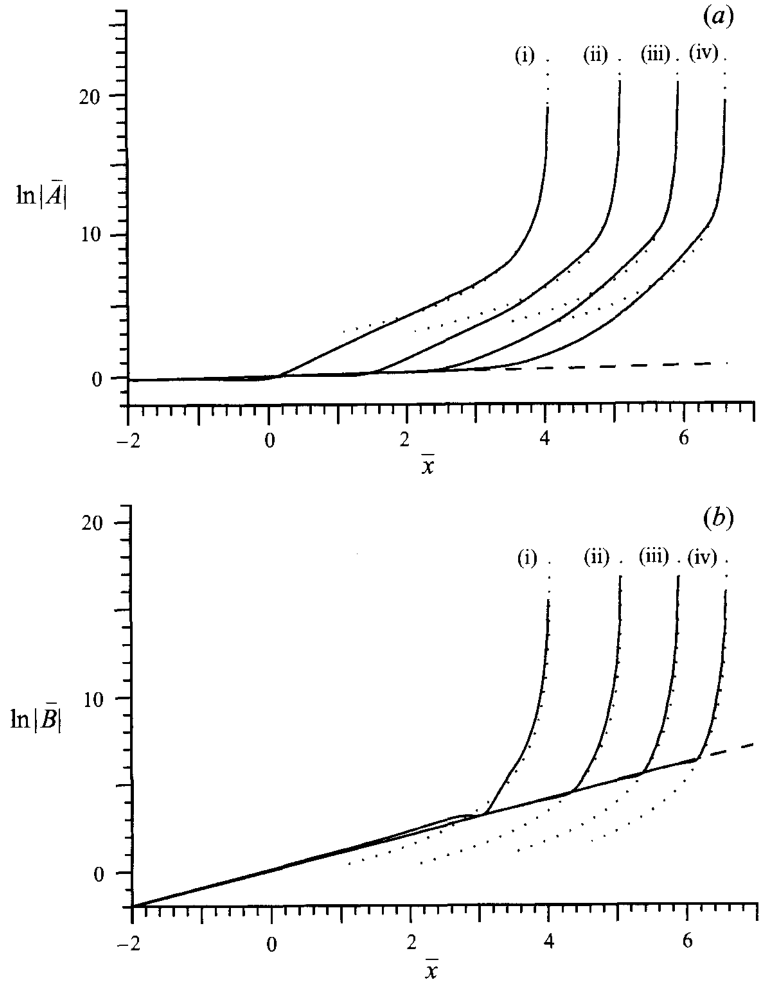

FIGURE 5. (*a*) Ln$|\bar{A}|$ *vs.* $\bar{x}$ and (*b*) ln$|\bar{B}|$ *vs.* $\bar{x}$ for $\chi_0 = 0.01$: (i) $\bar{\lambda} = 0$, (ii) $\bar{\lambda} = 5$, (iii) $\bar{\lambda} = 30$, and (iv) $\bar{\lambda} = 100$. The dotted lines represent the local singular solution (7.22) and the dashed line exponential growth.

can come into play either as a result of the super-exponential growth, or when the oblique modes are preferentially excited by some means. We first consider the case where a planar mode interacts with a single oblique mode. The rescaled equations for this case correspond to (5.34) and (5.35) with $\bar{\Upsilon}_a$ set to zero, and $\bar{\Upsilon}_p$ replaced by $\bar{\Upsilon}_p/2$. In the inviscid limit ($\bar{\lambda} = 0$), the development of the oblique and planar modes is displayed in figures 4(*a*) and 4(*b*) respectively for different sizes of χ_0. The curve (v) corresponds to $\chi_0 = 0$, and is included as a reference. It is seen that for any non-zero χ_0, the amplitudes of both the planar and oblique modes develop a singularity within a finite distance due to the two-way coupling between them. For small χ_0 (see e.g. curves (iii) and (iv) in figure 4*a*), the oblique-mode amplitude follows curve (v) over a considerable distance, indicating that the system experiences a prolonged secondary-instability stage before terminating in the finite-distance singularity. For relatively large χ_0, the secondary instability stage is by-passed, as shown by curves (i) and (ii). This is because in this case, the initial amplitude of the oblique mode is large enough

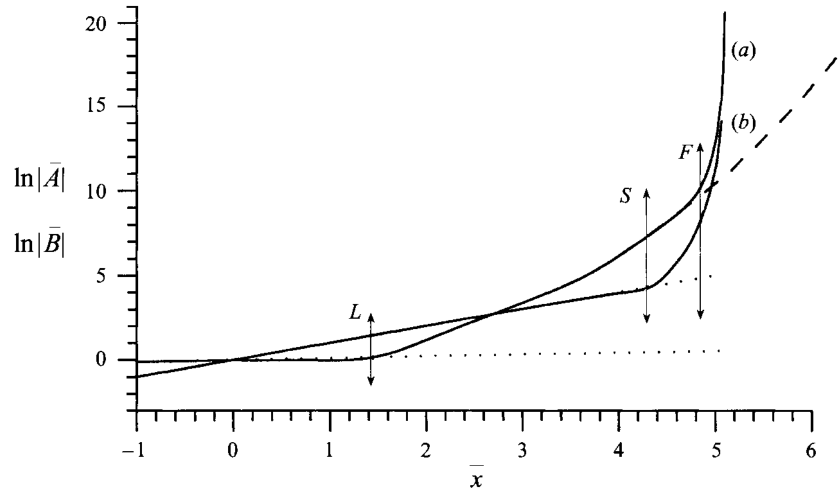

FIGURE 6. $\text{Ln}|\bar{A}|$ (curve a) and $\ln|\bar{B}|$ (curve b) *vs.* $\bar{x}$ for $\chi_0 = 0.01$ and $\bar{\lambda} = 5$. The dashed line represents the 'secondary-instability solution' and the dotted lines the exponential growth.

to cause the planar mode to deviate from exponential growth at the beginning of the interaction (see e.g. curves (i) and (ii) in figure 4b), and this in turn affects the development of the oblique mode itself.

In figure 5(a,b), we plot the development of the oblique and planar modes for different sizes of $\bar{\lambda}$. It is apparent that viscous effects postpone the formation of the finite-distance singularity. However, viscosity can neither eliminate the singularity, nor alter its final structure, no matter how large the value of $\bar{\lambda}$.

In order to illustrate the roles of the phase-locked modal interaction and the cubic feedback interaction, the amplitudes of the oblique and planar modes are plotted together in figure 6. Four distinct stages can be identified. Up to L is the linear stage where no appreciable interaction takes place, and so the modes evolve exponentially. From L to S, the oblique mode departs from exponential growth, and follows the secondary-instability development represented by the dashed curve. The planar mode still follows the linear theory closely up to S, indicating that the feedback effect of the oblique mode on the planar mode is negligible. Thus the evolution between L and S may be regarded as a secondary-instability stage. Starting from S, the oblique mode becomes sufficiently large to produce a feedback effect on the planar mode, causing the latter to deviate from linear growth. However, this deviation is not immediately felt by the oblique mode, which continues to follow the secondary-instability development up to F, i.e. it evolves as if the planar mode were still growing exponentially. Finally, the amplitude of the oblique mode departs from the secondary-instability growth (the dashed line), and soon a finite-distance singularity is formed. This last régime is the fully coupled stage.

We now solve the full equations (5.34) and (5.35) for the case where a planar mode interacts with a pair of oblique modes. The development of the disturbance in the inviscid limit is shown in figure 7(a,b) for four different values of χ_0. For small but non-zero χ_0 (e.g. curve iii), the evolution of the oblique modes overlaps that for $\chi_0 = 0$ over a considerable distance, implying that the oblique modes evolve through the linear and secondary–instability stages and finally terminate in a finite-distance singularity. The singularity has essentially the same structure as that for a single oblique mode,

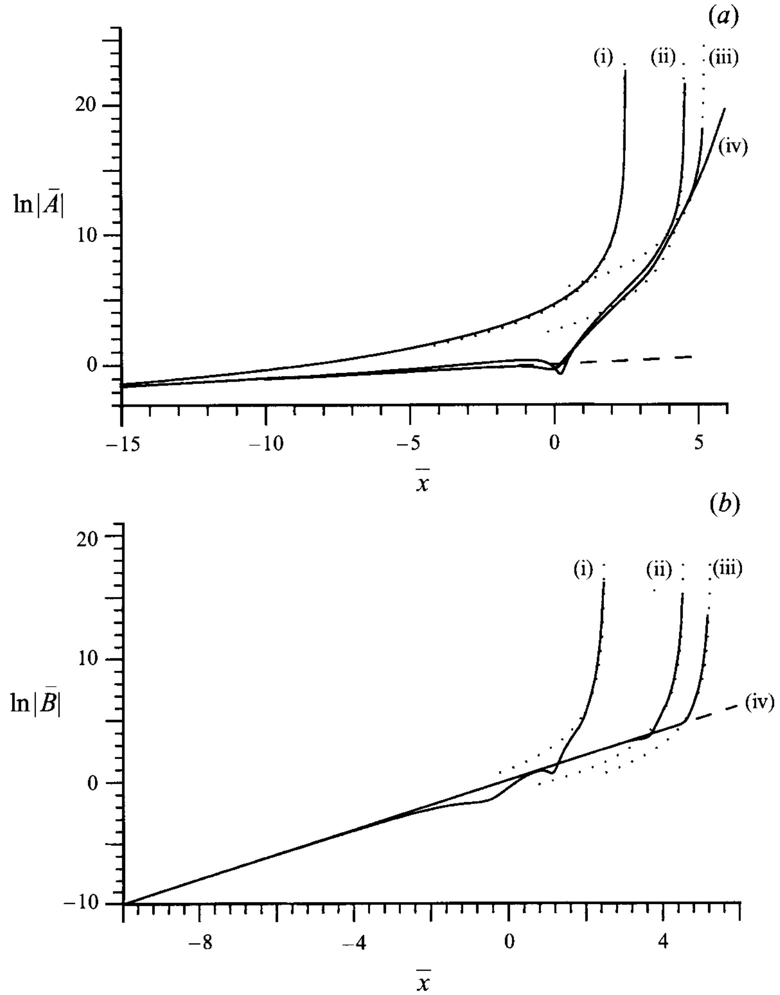

FIGURE 7. (*a*) $\text{Ln}|\bar{A}|$ *vs.* $\bar{x}$ and (*b*) $\ln|\bar{B}|$ *vs.* $\bar{x}$ for $\bar{\lambda} = 0$: (i) $\chi_0 = 10^{-3}$, (ii) $\chi_0 = 10^{-4}$, (iii) $\chi_0 = 10^{-5}$, and (iv) $\chi_0 = 0$. The dotted lines represent the local singular solution (7.22), and the dashed line the exponential growth.

although here the interaction between the oblique modes makes an extra contribution to its formation. For moderately large χ_0 (curve ii), the oblique modes do not evolve through a well-defined secondary-instability stage although the planar mode closely follows the linear theory over most of the evolution distance (see curve (ii) in figure 7*b*). The reason for the absence of such a stage is that at the end of the linear stage, the oblique modes have become sufficiently large that the phase-locked interaction and the self-interaction are comparable in strength. For even larger χ_0 (curve i in figure 7*a*), the self-interaction comes into play immediately after the linear stage and subsequently dominates. The secondary-instability stage is thus completely by-passed. However, given that the oblique modes have a much smaller linear growth rate, such a scenario is unlikely to occur unless they are preferentially excited.

Finally, as in the case of a single oblique mode, viscosity acts to delay the occurrence of the finite-distance singularity. This is demonstrated in figure 8(*a*,*b*).

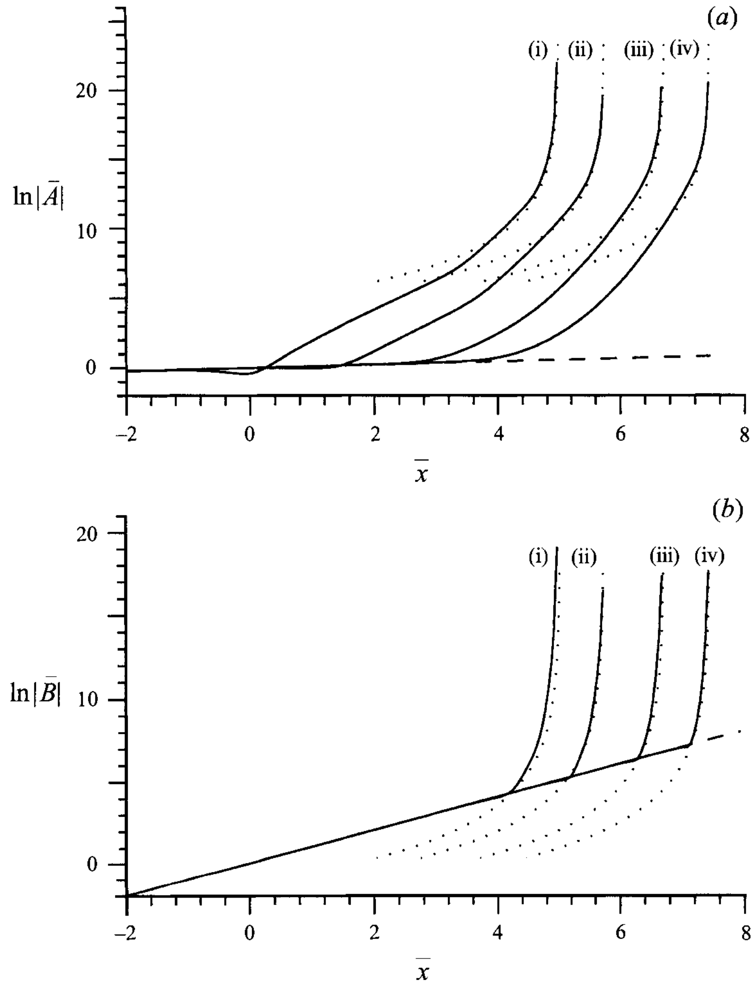

FIGURE 8. (*a*) Ln$|\bar{A}|$ *vs.* $\bar{x}$ and (*b*) ln$|\bar{B}|$ *vs.* $\bar{x}$ for $\chi_0 = 10^{-5}$: (i) $\bar{\lambda} = 0$, (ii) $\bar{\lambda} = 5$, (iii) $\bar{\lambda} = 50$, and (iv) $\bar{\lambda} = 200$. The dotted lines represent the local singular solution (7.22) and the dashed line the exponential growth.

8. Conclusions and discussion

This paper has studied the interactions between a planar mode and one or more oblique modes which share the same phase speed. This *phase-locked modal interaction* represents a new mechanism that can induce rapid growth of three-dimensional disturbances. Specifically, it is found that once the planar mode has attained a certain magnitude, it promotes enhanced, and ultimately super-exponential, growth of the oblique mode(s), As a result, the amplitude of the oblique mode(s) may soon overtake that of the planar mode, and the disturbance can evolve to a 'fully coupled' stage. Such a phase-locked interaction can operate under much less restrictive conditions than, say, the resonant triad, and so can have a wide application. The key condition for this mechanism to operate is phase-locking, i.e. the equality of phase speed. We note that in some experiments (e.g. Corke *et al.* 1992; Williamson & Prasad 1993*a*,*b*), the rapid growth of the oblique modes indeed coincides with the streamwise location at which this condition is satisfied. The fact that rather regular (three-dimensional) flow structures are observed is also an indication that the dominant disturbances are phase-locked, since otherwise the changing phase relation would prevent the formation of a regular pattern.

It is interesting to compare the phase-locked interaction with the resonant triad and the side-band instability, all of which can lead to the rapid (super-exponential) amplification of certain three-dimensional disturbances. As we have emphasized, the phase-locked interaction takes place between two modes, whereas the other two mechanisms involve a three-mode coupling. Nevertheless, the resonant triad can be regarded as a singular case of the phase-locked interaction in which the forced difference mode happens to coincide with an eigenmode. This most commonly arises when the oblique mode is a subharmonic of the planar mode, propagating at 60° to the flow direction, since then the forced difference mode is also an oblique subharmonic, propagating at −60°. Goldstein & Lee (1992) and Wundrow *et al.* (1994) show that the resulting resonant-triad interaction can promote super-exponential growth of the oblique modes. On the other hand, there are situations where subharmonic resonance is inactive, notably for sinuous modes in a symmetric plane wake or jet. In that case, however, a difference mode may coincide with a varicose eigenmode, thus giving rise to an active resonant triad of mixed modes. This has been analysed by Wu (1996), who finds that super-exponential growth is once again possible. For both types of triad, the phase-locked interaction can be recovered in a limiting case of strong detuning.

The side-band instability, by contrast, is a quite separate type of nonlinearity which operates in the weakly three-dimensional limit. Specifically, the cubic interaction between a planar mode $(\alpha_0, 0, c)$ and a weakly oblique 'side-band' mode $(\alpha, -\beta, c)$ gives rise to a forced mode $(2\alpha_0 - \alpha, \beta, c)$. When $\beta \ll 1$ and $\alpha \approx \alpha_0$, this combination mode approximately coincides with an eigenmode (the second 'side-band'), and the three modes can participate in a quasi-resonant interaction. In a future publication we shall show that this mechanism is distinct from, and operates alongside, the two-mode phase-locked interaction, and the two effects are comparable for a certain critical obliqueness. The phase-locked interaction dominates for larger obliqueness, the side-band resonance for smaller. An investigation of the latter régime shows that the side-bands can grow super-exponentially. Taken together, these various mechanisms may help to explain why three-dimensional disturbances, whether subharmonics or not, become dominant in the later stage of transition. Since the phase-locked interaction is less selective than the other mechanisms, it may play some part in spectral broadening and randomization of the flow (e.g. Corke & Mangano 1989).

Another type of interaction which has been proposed to explain the development of three-dimensional structure, especially the mean-flow distortion, involves merely a pair of oblique modes (α, β, c) and $(\alpha, -\beta, c)$, and no planar mode. This has been studied in the context of non-equilibrium critical layers by Goldstein & Choi (1989) and Wu *et al.* (1993). Such an interaction comes into play when the oblique modes have a magnitude of $O(\mu^3)$, where μ is the order of the growth rate. Given that for incompressible flows the fastest growing mode is two-dimensional, it seems more likely that the small oblique modes first attain the necessary magnitude as a result of interaction with a planar mode, either through a resonant-triad interaction, or through the phase-locked interaction, or through side-band interaction. This may then lead to the fully coupled stage where both the mutual interaction between the oblique waves and their feedback on the planar mode become important. On the other hand, the interaction between a pair of oblique modes may be the first nonlinear activity to take place if they have a larger linear growth rate; this can be the case for supersonic boundary layers of low Mach number (Leib & Lee 1995) and for flows with pre-existing streamwise vortices (Goldstein & Wundrow 1995).

In this paper, we have paid particular attention to the case of a plane wake. For this flow, it appears that the enhanced growth of sinuous three-dimensional disturbances cannot be attributed to subharmonic resonance, which is inactive, but may rather be the result of a phase-locked interaction. This conclusion seems to be strongly supported by the experimental work of Williamson & Prasad (1993*a*,*b*), which appeared when our theoretical study was in progress. They observed that an oblique shedding wave, of frequency f_K say, can interact with the dominant two-dimensional mode of frequency f_T (which is excited by some free-stream disturbance) to seed an oblique mode with the frequency $(f_K - f_T)$ in the far wake. Williamson & Prasad found that the oblique mode soon overtook the two-dimensional mode and became dominant. In the experiments, the shedding wave had a frequency much higher than those of the locally unstable modes, and decayed exponentially (see also Cimbala, Nagib & Roshko 1988). It did not appear to participate in any further nonlinear interactions after acting to trigger the oblique mode $(f_K - f_T)$. Therefore the interaction appears to be between two waves: the two-dimensional mode f_T and the oblique mode $(f_K - f_T)$. We believe that the enhanced growth of the oblique mode arises from a phase-locked interaction between these two modes, since they were found to have the same phase speed in the experiments. Williamson & Prasad also noted that increasing the magnitude of the two-dimensional wave led to enhanced growth of the oblique mode. This is predicted by our theory since equation (5.27) shows that the strength of the phase-locked interaction is proportional to $|B|^2$.

A different set of experiments was carried out by Corke *et al.* (1992). They introduced a two-dimensional wave simultaneously with a pair of oblique subharmonics, and mapped out the detailed development of each wave. Again, enhanced growth of the oblique modes was observed. They interpreted this in terms of subharmonic parametric resonance and a secondary-instability theory of Floquet type. However, since the measured growth rate of the oblique modes, though substantially larger than that predicted by linear theory, is comparable to that of the planar mode, it is not justifiable to treat the planar mode as quasi-steady in the secondary-instability calculations. For this and other reasons given in the introduction, we believe that the phase-locked interaction, which takes proper account of the non-equilibrium nature of the planar wave, is a more viable mechanism. This might be verified experimentally by varying the initial phase difference between the planar and the oblique waves. If this were found not to affect the development of the disturbance, that would favour the phase-locked interaction mechanism over the subharmonic resonance. The case *II* studied in this paper is similar to the experimental conditions of Corke *et al.* (1992). In our computations, we have taken the velocity defect parameter to be comparable with that in these experiments, and the obliqueness angle to be 60°, so that the oblique modes are (sinuous) subharmonics of the (sinuous) planar mode. As noted above, the development of the oblique waves in the 'secondary-instability' régime is in agreement with the measurement. Corke *et al.* (1992) also observed that while the distribution of the vertical velocity of the subharmonic is initially symmetric, as characteristic of a sinuous mode, it becomes increasingly asymmetric further downstream. This may be explained by the fact that for 60° obliqueness, the forced difference mode also has the subharmonic frequency, and is varicose in character.

Other experiments (e.g. Sato 1970; Sato & Saito 1975, 1978; Miksad *et al.* 1982) also showed a significantly large difference mode when a plane wake is excited by two frequencies. This is consistent with our theoretical prediction. However, in those experiments the disturbance was believed to be two-dimensional while in our theory three-dimensionality plays an important role in inducing the large difference mode. It may be that a certain degree of three-dimensionality was present in the experiments.

The mechanism described in this paper applies to any quasi-parallel shear flow which supports Rayleigh instabilities. As will be shown in a forthcoming paper, a similar theory may be constructed for Tollmien–Schlichting instabilities in, say, boundary layers. We further observe that the phase-locked interaction can also take place between two oblique modes which have the same phase speed but different frequencies: the mode with the higher frequency can accelerate the development of the lower-frequency mode when the amplitude of the former reaches $O(\mu^{7/2})$. In an incompressible flow, for which the fastest growing linear instability is two-dimensional, a planar mode may well be the first to attain this amplitude. On the other hand, the phase-locked interaction between two oblique waves may be important to transition in supersonic flows where linear theory predicts that the fastest growing mode is three-dimensional.

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Appendix A

The kernels appearing in (4.52), and in the amplitude equations for the planar mode, are specified by

$$
\begin{aligned}
K_b(\xi,\eta\,|\,\Lambda) = {} & v_0^3\xi(\xi+\eta)(v_0\xi+v_d\eta)\mathrm{e}^{-\Lambda\sigma_d^{-1}(v_0\xi+v_d\eta)^3+\Lambda v_0\sigma_d^{-1}\xi^3} \\
& +v_0^3\sin^2\theta\,\xi(\xi+\sigma_d\eta)(\sigma_s\xi+\eta)\mathrm{e}^{\Lambda(1-v_d^{-1})\xi^3-\Lambda\sigma_d^{-1}(v_0\xi+v_d\eta)^3} \\
& -v_0^3\sin^2\theta\int_0^{\xi}(\xi+\sigma_d\eta)\big[(v_0\xi+v_s\eta+2\sigma_s\gamma)-6\Lambda\gamma^3(\xi+\eta+\sigma\gamma)\big] \\
& \qquad\times\mathrm{e}^{-\Lambda\gamma^3-\Lambda\sigma_d^{-1}(v_0\xi+v_d\eta)^3+\Lambda v_d^{-1}(v_0\xi-v_d\gamma)^3-\Lambda v_0^2(\xi-\gamma)^3}\,\mathrm{d}\gamma\,,
\end{aligned}
\tag{A 1}
$$

$$
\begin{aligned}
K_c(\xi,\eta\,|\,\Lambda) = {} & v_0^4\xi^3\mathrm{e}^{-\Lambda v_0^2v_s\xi^3-3\Lambda v_0^2\xi^2\eta} \\
& +v_0\sin^2\theta(\xi+\eta)(v_0\xi+\eta)(v_s\xi+\eta)\mathrm{e}^{-\Lambda(1-v_d^{-1})(\xi+\eta)^3-\Lambda\sigma_d^{-1}(v_0\xi+\eta)^3} \\
& -v_0\sin^2\theta\,\eta(\xi+\eta)(v_s\xi+\eta)\mathrm{e}^{-\Lambda\sigma_s^{-1}\eta^3-\Lambda(1+v_s^{-1})(v_s\xi+\eta)^3} \\
& +v_0^2\sin^2\theta\,\xi\eta(v_0\xi+\eta)\mathrm{e}^{-\Lambda\eta^3-\Lambda(v_0\xi+\eta)^3-\Lambda v_0^2\xi^3} \\
& -v_0^2\sin^2\theta\int_0^{\xi}\Big[[v_0\xi+(1+2\sigma)\eta+2\sigma_s\gamma]-6\Lambda\sigma(\eta+\gamma)^3(v_0\xi+\eta+\gamma)\Big] \\
& \qquad\times(v_0\xi+\eta)\mathrm{e}^{-\Lambda(\eta+\gamma)^3-\Lambda\sigma_d^{-1}(v_0\xi+\eta)^3+\Lambda v_d^{-1}(v_0\xi+\eta-v_d\gamma)^3-\Lambda v_0^2(\xi-\gamma)^3}\,\mathrm{d}\gamma \\
& +v_0^2\sin^2\theta\int_0^{\xi}\eta\Big[[2\xi+(2\sigma-1)\eta-2\sigma_d\gamma]-6\Lambda\sigma(\eta+\gamma)(v_0\xi+\eta+\gamma)^3\Big] \\
& \qquad\times\mathrm{e}^{-\Lambda(v_0\xi+\eta+\gamma)^3-\Lambda\sigma_s^{-1}\eta^3-\Lambda v_s^{-1}(\eta+v_s\gamma)^3-\Lambda v_0^2(\xi-\gamma)^3}\,\mathrm{d}\gamma\,.
\end{aligned}
\tag{A 2}
$$

Appendix B

The kernel K_a associated with the self-interaction of the oblique modes is given by (3.85) of Wu *et al.* (1993). With a slight adjustment of notation, it takes the form

$$
\begin{aligned}
&K_a(\xi,\eta\,|\,\Lambda) = \tilde{K}^{(0)}(\xi,\eta)(2\xi^3+\xi^2\eta) \\
&+2\sin^2\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\eta[\xi^2+2\xi(\eta-\zeta)]\mathrm{e}^{-2\Lambda\zeta^3-3\Lambda\xi\zeta^2}\,\mathrm{d}\zeta\right. \\
&+\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi[\zeta(2\eta+3\zeta)-\xi(\xi+2\eta+2\zeta)]\mathrm{e}^{-3\Lambda\xi\zeta^2}\,\mathrm{d}\zeta \\
&+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi\eta\zeta[1+6\Lambda(\xi-\zeta)(\xi+\eta+\zeta)^2]\Pi_0(\xi,\eta,\zeta)\,\mathrm{d}\zeta \\
&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi[(\eta+\zeta)(\eta+3\zeta)-(\xi+\eta)(\xi+\eta+2\zeta)]\mathrm{e}^{-3\Lambda(\xi+\eta)(2\eta+\zeta)\zeta}\,\mathrm{d}\zeta\right\} \\
&+8\sin^4\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi\mathrm{d}\zeta\,\mathrm{e}^{-3\Lambda\xi\zeta^2}\int_0^{\eta+\zeta}(\upsilon-\eta-\zeta)[1+6\Lambda(\xi-\zeta)\zeta^2]\mathrm{e}^{-\Lambda(2\upsilon^3+3\xi\upsilon^2)}\,\mathrm{d}\upsilon\right. \\
&+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi\mathrm{d}\zeta\,\Pi_0(\xi,\eta,\zeta)\int_0^\zeta(\zeta-\upsilon)[1+6\Lambda(\xi-\zeta)(\xi+\eta+\zeta)^2]\mathrm{e}^{\Lambda(2\upsilon^3+3\eta\upsilon^2)}\,\mathrm{d}\upsilon \\
&\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi\mathrm{d}\zeta\,\mathrm{e}^{-3\Lambda(\xi+\eta)(2\eta+\zeta)\zeta}\int_0^\zeta(\upsilon-\zeta)[1+6\Lambda(\xi-\zeta)(\eta+\zeta)^2]\mathrm{e}^{-\Lambda[2\upsilon^3+3(\xi+\eta)\upsilon^2]}\,\mathrm{d}\upsilon\right\},
\end{aligned}
\tag{B 1}
$$

where the auxiliary kernels $\tilde{K}^{(0)}$, $\tilde{K}^{(1)}$ and Π_0 are defined by

$$\tilde{K}^{(0)}(\xi,\eta)=\mathrm{e}^{-\Lambda(2\xi^3+3\xi^2\eta)}, \quad \tilde{K}^{(1)}(\xi,\eta)=\mathrm{e}^{-\Lambda[\xi^3+\eta^3+(\xi+\eta)^3]}, \tag{B 2}$$

$$\Pi_0(\xi,\eta,\zeta)=\mathrm{e}^{-\Lambda(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)}. \tag{B 3}$$

Although the kernel K_a is algebraically complicated, in the inviscid limit ($\Lambda=0$) it simplifies to the form

$$K_a(\xi,\eta\,|\,0)=(2\xi^3+\xi^2\eta)-2\sin^2\theta(2\xi^3-\xi\eta^2)-4\sin^4\theta(\xi^2\eta+\xi\eta^2), \tag{B 4}$$

which was obtained by Goldstein & Choi (1989).

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